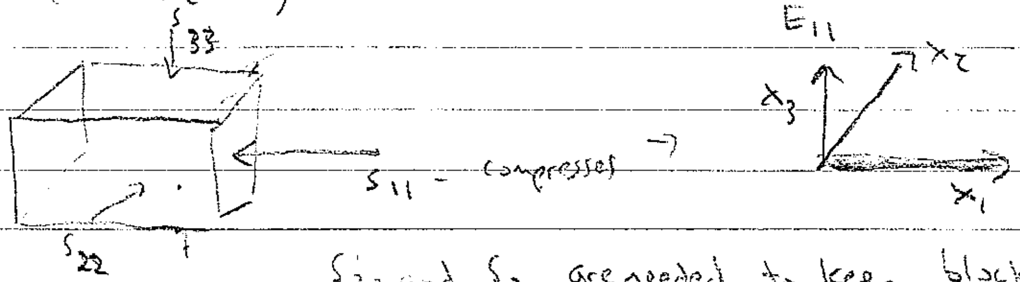
MATH 323 Assignment 4 Tutorial 5 Sketch Notes

1. A specimen of isotropic material is subjected to compression S_{11} but is constrained so that it cannot expand in the x_2 and x_3 directions. Show that the apparent modulus of elasticity is:

$$Y' = \frac{Y(1-\nu)}{(1+\nu)(1-2\nu)}$$

where from the notes in Assignment 4 we note: $Y \equiv \frac{S_{11}}{E_{11}}$ when S_{22} and $S_{33}=0$



S_{22} and S_{33} are needed to keep block from expanding in x_2 and x_3 directions

Also ν is defined as Poisson's ratio. $\nu = -\frac{E_{22}}{E_{11}}$

So our apparent strain tensor is $E = \begin{pmatrix} E_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ since $\frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial x_3} = 0$

Only change is in x_1 direction.

So the apparent modulus $Y' = \frac{S_{11}'}{E_{11}'} = \frac{S_{11}}{E_{11}}$

In the isotropic case (which we have) $S_{ij} = \lambda \delta_{ij} E_{kk} + 2\mu E_{ij}$

$E_{kk} = E_{11}$ since $E_{22} = E_{33} = 0$

$S_{11} = (\lambda + 2\mu) E_{11}$ — consistent w/ P-wave propagation speed $\sqrt{\frac{\lambda+2\mu}{\rho}}$

$S_{22} = \lambda E_{11} + 2\mu E_{22} = \lambda E_{11}$ ⇒ this is the stress needed to keep from expanding in E_{22} + E_{33}

$S_{33} = \lambda E_{11} + 2\mu E_{33} = \lambda E_{11}$

$E_{12} = E_{21} = E_{23} = E_{32} = 0$ so $S_{12} = 0 = S_{13} = S_{23}$

Apparent modulus $= \frac{S_{11}}{E_{11}} = \lambda + 2\mu = Y'$

Now, from Assignment 4, $Y = \frac{\mu(3\lambda + 2\mu)}{(\lambda+\mu)}$ $\qquad \nu = \frac{\lambda}{2(\lambda+\mu)}$

P.2. Tutorial 5 Q1

We want to show that $\frac{Y(1-\nu)}{(1+\nu)(1-2\nu)} = \lambda + 2\mu$

$$\frac{Y(1-\nu)}{(1+\nu)(1-2\nu)} = \frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)} \frac{\left(1-\frac{\lambda}{2(\lambda+\mu)}\right)}{\left(1+\frac{\lambda}{2(\lambda+\mu)}\right)\left(1-\frac{2\lambda}{2(\lambda+\mu)}\right)} \frac{(2(\lambda+\mu))}{(2(\lambda+\mu))}$$

$$= \frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}\left[\frac{2\lambda+2\mu-\lambda}{\left(1+\frac{\lambda}{2(\lambda+\mu)}\right)(2(\lambda+\mu)-2\lambda)}\right] = \frac{\mu(3\lambda+2\mu)(\lambda+2\mu)}{(\lambda+\mu)\left(1+\frac{\lambda}{2(\lambda+\mu)}\right)(2\mu)}$$ ✓

$$= \frac{\mu(3\lambda+2\mu)(\lambda+2\mu)}{\mu(2(\lambda+\mu)+\lambda)} = \frac{(3\lambda+2\mu)(\lambda+2\mu)}{(3\lambda+2\mu)} = \lambda+2\mu$$ QED

Tutorial 5 - Assignment 4 - Q1 Part 2

Show that we can re-write Hooke's Law for isotropic elastic solids as: $E_{ij} = \frac{(1+\nu)}{Y} S_{ij} - \frac{\nu}{Y} S_{kk}\delta_{ij}$

Start w/ standard eqn: $S_{ij} = \lambda \delta_{ij} E_{kk} + 2\mu E_{ij}$ (2) in notes on p. 19

rearrange:

$$E_{ij} = \frac{S_{ij}}{2\mu} - \frac{\lambda}{2\mu}\delta_{ij} E_{kk}$$

Also, get S_{kk} in terms of E_{kk}: from Eqn 2:

$S_{kk} = 3\lambda E_{kk} + 2\mu E_{kk}$ (since E_{kk} contributes to all of $\delta_{11}, \delta_{22}, \delta_{33}$)

$$S_{kk} = (3\lambda + 2\mu) E_{kk}$$

$$\therefore E_{ij} = \frac{S_{ij}}{2\mu} - \frac{\lambda}{2\mu(3\lambda+2\mu)} S_{kk}\delta_{ij}$$

So we need to show: (1) that $\frac{1+\nu}{Y} = \frac{1}{2\mu}$ and 2): $\frac{\nu}{Y} = \frac{\lambda}{2\mu(3\lambda+2\mu)}$

$$\frac{1+\nu}{Y} = \frac{1 + \frac{\lambda}{2(\lambda+\mu)}}{\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}} = \frac{1+\frac{\lambda}{2(\lambda+\mu)}}{\mu(3\lambda+2\mu)}\frac{2(\lambda+\mu)}{2} = \frac{2\lambda+2\mu+\lambda}{2\mu(3\lambda+2\mu)} = \frac{(3\lambda+2\mu)}{2\mu(3\lambda+2\mu)} = \frac{1}{2\mu}$$

QED (1)

$$(2): \frac{\nu}{Y} = \frac{\frac{\lambda}{2\lambda+2\mu}}{\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}} \frac{(2(\lambda+\mu))}{2(\lambda+\mu)} = \frac{\lambda}{2\mu(3\lambda+2\mu)}$$

QED (2)

Tutorial Exc Problems:

2. If E_{kl} is defined $E_{12} = E_{21} = -2$, $E_{kl} = 0$ otherwise, evaluate

$$I = \int_0^{E_{kl}} u_{ij}\, du_{ij}$$

for each element

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{u_{ij}=0}^{u_{ij}=E_{kl}} u_{ij}\, du_{ij} = \frac{1}{2} u_{ij}^2 \Big|_0^{E_{kl}}$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{1}{2} u_{ij}^2 \Big|_0^{E_{kl}}$$

$$= \frac{1}{2}\left(u_{11}^2 + u_{12}^2 + u_{13}^2 + u_{21}^2 + u_{22}^2 + u_{23}^2 + u_{31}^2 + u_{32}^2 + u_{33}^2 \right)$$

replace $u_{ij} \to E_{ij} \Rightarrow$

$$= \frac{1}{2}\left(\underset{-0}{0} + \underset{-0}{4} + 0 + \underset{-0}{4} + 0 + 0 + 0 + 0 + 0 \right)$$

$$I = 4$$

3. Show that $u_2 = A \sin(wt \pm \nu x_1)$; $u_1 = 0$; $u_3 = 0$

is a solution of Navier's Equation without body forces

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + (\mu + \lambda) \frac{\partial^2 u_k}{\partial x_k \partial x_i}$$

provided $c = \frac{w}{\nu}$ satisfies $c^2 = \frac{\mu}{\rho}$

Proof by substitution:

$$\frac{\partial u_1}{\partial t} = 0 = \frac{\partial^2 u_1}{\partial t^2} = \frac{\partial^2 u_3}{\partial t^2} \quad ; \quad \frac{\partial u_2}{\partial t} = wA\cos(wt \pm \nu x_1)$$

$$\frac{\partial^2 u_2}{\partial t^2} = -w^2 A \sin(wt \pm \nu x_1)$$

$$\frac{\partial u_i}{\partial x_j \partial x_j} = 0 \quad \text{for } i = 2 \text{ and } i = 3$$

$\frac{\partial u_k}{\partial x_i} = 0$ for $k = 1, 3$, and $i = 2, 3$

$\frac{\partial u_2}{\partial x_j} = 0$ except for $j = 1$ the only non-zero is

$\frac{\partial u_2}{\partial x_1} = \nu A \cos(\omega t \pm \nu x_1)$

but $\frac{\partial^2 u_2}{\partial x_2 \partial x_1} = 0$

$$\frac{\partial u_2}{\partial x_1} = \pm \nu A \cos(\omega t \pm \nu x_1)$$

$$\frac{\partial^2 u_2}{\partial x_1 \partial x_1} = \mp \nu^2 A \sin(\omega t \pm \nu x_1)$$

So: Eqn being: u_2 only:

$$\rho\left(-\omega^2 A \sin(\omega t \pm \nu x_1)\right) = \mu \frac{\partial^2 u_2}{\partial x_2 \partial x_2} = \mu\left(\mp \nu^2 A\right)\left(\sin \omega t \pm \nu x_1\right)$$

Which has non-trivial solution for all t, x_1, if:

$$-\omega^2 \rho \cancel{A} = \mu\left(\mp \nu^2 \cancel{A}\right)$$

$$\mu \nu^2 = \omega^2 \rho \quad \text{or} \quad \frac{\omega^2}{\nu^2} = \frac{\mu}{\rho} \quad \text{QED}$$