Notes for Swing High Module

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MATH321/2/3

These notes provide supporting information that will likely be useful for students participating in the Swing High module.

1 The Simple Pendulum

This is an application of perturbation theory, based on Lin & Segel (**LS**) pp.48–55 (*Mathematics Applied to Deterministic Problems in the Natural Sciences*, by CC Lin and LA Segel, Classics in Applied Mathematics, SIAM, 1988).

The pendulum provides us with a simple system that is easy to visualise and that we have some physical feeling for, but which also is nonlinear and useful as an example of the use of perturbation methods for studying solutions.

The usual standard treatment of a pendulum is to reduce it to simple harmonic motion, by linearising. We will see how perturbation theory extends and improves upon this approximation.

Consider a rigid massless rod as illustrated in fig. (1), with a point mass M at the lower end, free to rotate in one dimension about a pivot point. Already we have idealised the actual pendulum, to simplify the problem. Intuition

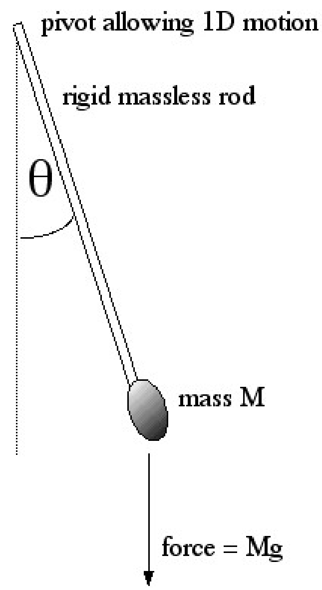


Figure 1: The simple pendulum

and personal experience (or professional advice) usually play an important part of these first steps in modelling.

To obtain a differential equation for the movement of this pendulum, we use Newton's law of motion, force equals mass times acceleration, $\mathbf{F} = M\mathbf{g}$, where force and acceleration are vector quantities (in bold font), and the magnitude of $\mathbf{g}$ is taken to be 9.8 ms^{-2}.

The force on the pendulum may be resolved into a component in the direction of the rod, which is balanced by tension in the rod and by the pivot point, and a component normal to the rod, which causes the mass to accelerate (see fig. (2)).

ASIDE: Velocity and Acceleration in polar coordinates

Because we know the rod moves according to the component in the θ direction, it is best to use polar coordinates here. Here is a brief review of

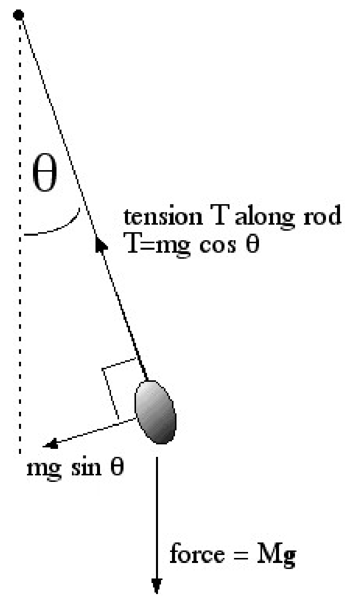


Figure 2: Forces on the simple pendulum

the general case with varying r and θ. Recall that if the cartesian unit vectors are $\mathbf{i}$ in the x-direction and $\mathbf{j}$ in the y-direction, then polar unit vectors are $\mathbf{i}_r = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$ in the radial direction and $\mathbf{i}_\theta = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$ in the (positive) θ direction (that is, tangent to a circle centered on origin and through the position of a particle (here, the mass M)).

Then the position vector with polar coordinates (r, θ) may be written in terms of the polar unit vectors as

$$\mathbf{r} = r\mathbf{i}_r \ .$$

Note that the angle information is contained in $\mathbf{i}_r$. In particular, note that taking derivatives of the definitions gives

$$\frac{d\mathbf{i}_r}{dt} = \dot{\theta}\mathbf{i}_\theta$$

where $\dot{\theta}$ means $d\theta/dt$, and

$$\frac{d\mathbf{i}_\theta}{dt} = -\dot{\theta}\mathbf{i}_r \ .$$

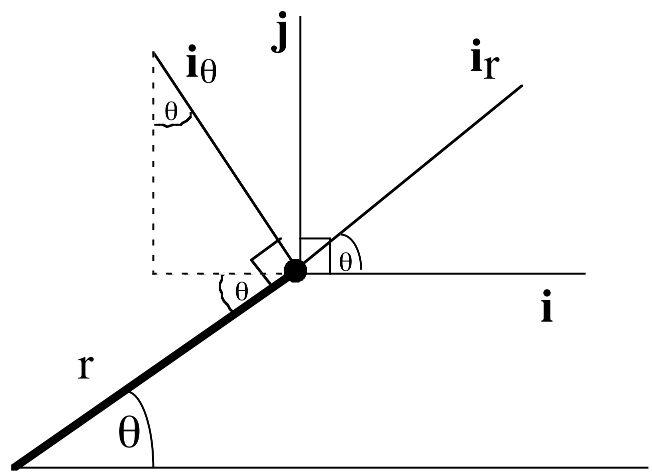


Figure 3: Polar coordinates and unit vectors

Then velocity is

$$\begin{aligned}\mathbf{v} &\equiv \frac{d\mathbf{r}}{dt} = \frac{d(r\mathbf{i}_r)}{dt} \\ &= \frac{dr}{dt}\mathbf{i}_r + r\frac{d\mathbf{i}_r}{dt} \\ &= \dot{r}\mathbf{i}_r + r\dot{\theta}\mathbf{i}_\theta\end{aligned}$$

and acceleration is

$$\mathbf{a} \equiv \frac{d\mathbf{v}}{dt} = (\ddot{r} - r\dot{\theta}^2)\mathbf{i}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{i}_\theta \ .$$

Back to the pendulum:

Then for our pendulum, with a fixed length $r = L$, the components of force and mass times acceleration in the direction $\mathbf{i}_\theta$ balance to give

$$-Mg\sin\theta = ML\ddot{\theta} \ .$$

Hence

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0\ . \tag{1}$$

We are interested in the solution with initial conditions

$$\theta(0) = a \qquad \dot{\theta}(0) = \Omega\ . \tag{2}$$

This completes what is often the most difficult part of mathematical modelling, taking a problem and reducing it to an equation to be solved, usually a differential equation. We now consider solving the mathematical problem, numerically and/or analytically. Analytical solutions will often involve asymptotic methods to obtain approximate analytic results.

1.1 Approximate analytic solutions

The simplest approach is to assume θ is small enough that $\sin\theta \sim \theta$, and obtain the linearised equation

$$\ddot{\theta} + \frac{g}{L}\theta = 0\ , \quad \theta(0) = a\ ,\ \dot{\theta}(0) = \Omega\ . \tag{3}$$

Solutions may be found by substituting an exponential form, to obtain the general solution

$$\theta_0 = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$$

where $\omega_0 = \sqrt{g/L}$. The initial conditions give

$$\begin{aligned} \theta_0(0) &= A + B = a\ , \\ \dot{\theta}_0(0) &= i\omega_0 A - i\omega_0 B = \Omega\ , \end{aligned}$$

which can be solved for A and B, or using

$$e^{ix} = \cos x + i\sin x$$

we can write the solution to the initial value problem in the form

$$\theta_0 = a\cos(\omega_0 t) + \frac{\Omega}{\omega_0}\sin(\omega_0 t)\ .$$

This solution is periodic, with period

$$P_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{L}{g}}\ .$$

However, the exact period of the solution to eqn (1) is not P_0, and for example, if one wishes to know the exact time of maximum swing, the prediction that this time is nP_0 , $n = 1, 2, \ldots$ becomes very inaccurate for large times, or large n.

1.2 Asymptotic expansions

If we rewrite eqn (1) in the form

$$\ddot{\theta} + \omega_0^2\theta = \omega_0^2(\theta - \sin\theta) = \omega_0^2(\theta^3/3! - \theta^5/5! + \ldots)\ , \tag{4}$$

we see that the solution θ_0 to the linearised problem has been obtained by setting the right-hand side of eqn (4) to zero. This is justified by noting that the size of terms on the left-hand side is of the order of θ, while the size of the terms on the right-hand side is of the order of θ^3, which is much smaller than θ as $\theta \to 0$.

Note that the terminology *is of the order of* means roughly *is about the same size as*, but may be given a more formal meaning, as in Appendix 3.1 of LS.

Then an improved solution θ_1 might be obtained in an iterative process, by solving

$$\ddot{\theta_1} + \omega_0^2\theta_1 = \omega_0^2\theta_0^3/3!\ , \theta_1(0) = a\ ,\ \dot{\theta}_1(0) = \Omega\ , \tag{5}$$

and in principle this process might be repeated again and again, to further improve the (analytic) approximation to the solution θ. This iterative process is called the method of successive approximations. It works because at each step, the problem to be solved is a linear one.

Before solving, it is convenient to rescale. Let

$$\Theta(t) = \frac{\theta(t)}{\theta_m}$$

where

$$\theta_m \equiv \max_t |\theta(t)| .$$

Now $|\Theta(t)| \leq 1$, which is useful for estimating sizes. Then eqn (4) becomes

$$\ddot{\Theta}+\omega_0^2\Theta = \omega_0^2\left[\Theta - \frac{1}{\theta_m}\sin(\theta_m\Theta)\right] \approx \omega_0^2\theta_m^2\frac{\Theta^3}{3!} \ , \ \Theta(0) = a/\theta_m \ , \ \dot{\Theta}(0) = \Omega/\theta_m \ . \tag{6}$$

The right-hand side of eqn (6) is of the order of θ_m^2, and we will assume that this is much less than one. This is clearer if you rescale time by ω_0.

Now we consider the case that $\Omega = 0$, that is, releasing the pendulum from an initial angle $\theta(0) = a$, we see that $\theta_m = a$. Hence, to make θ_m small, we will require that $a \ll 1$. Then eqn (6) becomes

$$\ddot{\Theta}+\omega_0^2\Theta = \omega_0^2\left[\Theta - \frac{1}{a}\sin(a\Theta)\right] = \omega_0^2\sum_{n=1}^{\infty}(-1)^{n+1}a^{2n}\frac{\Theta^{2n+1}}{(2n+1)!} \ , \ \Theta(0) = 1 \ , \ \dot{\Theta}(0) = 0 \ . \tag{7}$$

We expand Θ as a power series in the small parameter a:

$$\Theta = \Theta(t,a) = \Theta^{(0)}(t) + a\Theta^{(1)}(t) + a^2\Theta^{(2)}(t) + \ldots$$

and substitute this into eqn (7) and equate coefficients of powers of a. Note that the initial conditions become

$$\Theta^{(0)}(0) = 1 \ , \ \dot{\Theta}^{(0)}(0) = 0 \ ,$$

and

$$\Theta^{(n)}(0) = 0 \ , \ \dot{\Theta}^{(n)}(0) = 0 \ , n \geq 1 \ .$$

Then equating coefficients gives

$\mathbf{a^0}$:

$$\ddot{\Theta}^{(0)} + \omega_0^2\Theta^{(0)} = 0$$

which is the linearised problem. The solution is

$$\Theta^{(0)} = \cos(\omega_0 t) \ .$$

a$^{\mathbf{1}}$:

$$\ddot{\Theta}^{(1)} + \omega_0^2 \Theta^{(1)} = 0$$

which (considering the initial conditions) has solution $\Theta^{(1)} = 0$.

a$^{\mathbf{2}}$:

$$\ddot{\Theta}^{(2)} + \omega_0^2 \Theta^{(2)} = \frac{\omega_0^2}{6}[\Theta^{(0)}]^3 = \frac{\omega_0^2}{6}\cos^3(\omega_0 t) = \frac{\omega_0^2}{24}[\cos(3\omega_0 t) + 3\cos(\omega_0 t)] \ .$$

The solution to this nonhomogeneous differential equation is

$$\Theta^{(2)} = \frac{1}{192}[\cos(\omega_0 t) - \cos(3\omega_0 t)] + \frac{\omega_0 t}{16}\sin(\omega_0 t) \ .$$

So far we have the following approximation to the solution:

$$\Theta = \cos(\omega_0 t) + a^2\left[\frac{1}{192}[\cos(\omega_0 t) - \cos(3\omega_0 t)] + \frac{\omega_0 t}{16}\sin(\omega_0 t)\right] + \dots$$

The second term is a correction to the amplitude, the third term is a higher harmonic, and the fourth term (which has the term t in it) grows without bound as $t \to \infty$. This is a problem, as it contradicts the assumption we have made in deriving this solution, that corrections involving the higher powers of a are small. Put another way, we constructed Θ to be no bigger than one in magnitude, but this third term gives an unbounded Θ, and the approximation fails for large times.

We are actually seeking an asymptotic expansion, with each term much less than the previous one (as $a \to 0$). The power series in a guarantees this, but for any small fixed a, the growth in time eventually destroys the usefulness of the expansion. This problem term is called a *secular term.* It is a sign that our approach needs to be improved.

The source of the problem lies in the fact that the period of the nonlinear pendulum is not exactly $2\pi/\omega_0$. In fact, you will show in your assignment that the exact period is

$$P = \frac{4}{\omega_0}\int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}}$$

where

$$k^2 = \sin^2(a/2) \ .$$

The period P approaches $P_0 = 2\pi/\omega_0$ as $a \to 0$, but is not equal to P_0.

The expansion we have used is inherently periodic with period P_0. Poincaré's method seeks to improve upon this restriction.

Poincaré's Method

This is also called the Poincaré — Lindstedt method, and is also to be found in Logan (p. 42)[1]. It is also related to the method of multiple scales (LS 11.2). We allow the period to change (a little) by also expanding time as a power series in a. Let

$$\Theta = \Theta^{(0)}(\tau) + a\Theta^{(1)}(\tau) + a^2\Theta^{(2)}(\tau) + \dots$$

and

$$t = \tau + at^{(1)}(\tau) + a^2 t^{(2)}(\tau) + \dots$$

and we choose τ so that secular terms are removed. To simplify calculations, in fact we take

$$t = \tau(1 + a^2 h_2 + \dots)$$

where h_2 is some constant to be determined. Then substituting and equating powers of a gives

$$\Theta^{(0)} = \cos(\omega_0 \tau)$$

as before, and

$$\frac{d^2\Theta^{(2)}}{d\tau^2} + \omega_0^2\Theta^{(2)} = \omega_0^2(1/8 - 2h_2)\cos(\omega_0\tau) + \frac{\omega_0^2}{24}\cos(3\omega_0\tau) \ .$$

The first term on the right-hand side gives rise to the secular term, and can be eliminated by choosing $h_2 = 1/16$. Then note that

$$\Theta^{(0)} = \cos(\omega_0\tau) \approx \cos[\omega_0 t(1 - a^2/16)] \ ,$$

[1] *Applied Mathematics - a contemporary approach*, JD Logan, Wiley 1987. KAIST library QA37.2.L64 1987

and we have found an approximate correction to the period,

$$P \sim \frac{2\pi}{\omega_0}\left(1+\frac{a^2}{16}\right) \ .$$

2 Phase Plane Analysis

A useful way to view and understand solution behaviour is the geometric approach of using a phase space. We illustrate with a phase plane analysis of the pendulum (LS 11.3).

Consider the differential equation

$$\ddot{\theta}(t)+\omega_0^2 \sin\theta(t) = 0 \ , \theta(0) = a \ , \ \dot{\theta}(0) = \Omega \tag{8}$$

and rescale time as $t^* = t\omega_0$. Then in terms of the new time t^*,

$$\frac{d^2\theta}{d(t^*)^2} + \sin\theta(t^*) = 0 \ ,$$

and we will drop the *'s for convenience from now on. So we consider for the nonlinear pendulum,

$$\ddot{\theta} + \sin\theta = 0 \ , \quad \theta(0) = a \ , \ \dot{\theta}(0) = b \equiv \Omega/\omega_0 \ . \tag{9}$$

To study solutions in the phase plane, we convert this second-order differential equation to two coupled first-order equations: let $\omega = \dot{\theta}$. Then

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -\sin\theta \ . \end{aligned} \tag{10}$$

Solutions are usefully viewed in the phase plane with θ along the x-axis and ω along the y-axis. We find level curves (along which solutions move) by deriving an equation expressing conservation of energy: multiplying eqn (9) through by $\dot{\theta}$, it follows that

$$\frac{d}{dt}\left(\frac{\dot{\theta}^2}{2} - \cos\theta\right) = 0 \ ,$$

so that

$$\left(\frac{\dot{\theta}^2}{2} - \cos\theta\right) = \text{constant} = b^2/2 - \cos a \ ,$$

that is,

$$\omega^2 = 2\cos\theta - 2\cos a + b^2 \ . \tag{11}$$

This equation implicitly defines (constant energy) curves in the phase plane, which solutions must lie on. A given set of initial conditions defines one set of curves which that solution must move along as time progresses.

For small θ, the linearised version of this equation is

$$\omega^2 + \theta^2 = b^2 - 2\cos a \ ,$$

that is, circles in the phase plane. Since $\dot{\theta} = \omega$, the direction of movement on the circles is clockwise.

Sketching of the more general nonlinear case is helped if you note that the equation is invariant under a change of sign of either ω or θ (so we can consider the first quadrant only), and that the right-hand side is periodic with period 2π, so we only need to consider $\theta \in (0, 2\pi)$. Combining these two observations allows us to only consider $\theta \in (0, \pi)$. Also remember that small θ values give circles. The phase portrait then looks like that in Fig. (4).

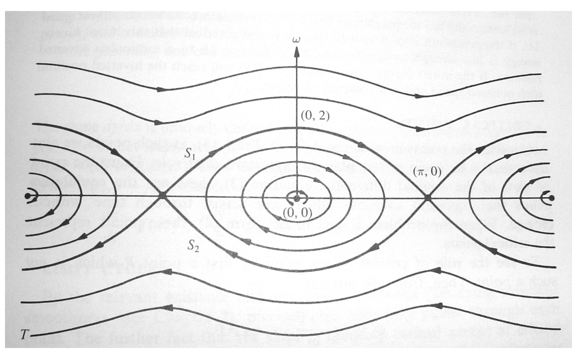


Figure 4: The Phase plane for the nonlinear pendulum

Note the closed loops (trajectories) corresponding to smaller energies or smaller angular velocities, the open trajectories that look like cosine curves corresponding to large energy, large angular velocities, propellor-like motion of the pendulum, and the special curves (called *separatrices*) that separate the two behaviours. Can you picture what a pendulum is doing, if the phase plane solution is on a separatrix?

Equilibrium Points
or *fixed points* or *critical points* are places in phase space where the velocity is zero, that is, the right-hand sides of the first-order coupled differential equations are simultaneously zero. For our problem, this means $\omega = 0$ and $\theta = n\pi\ ,\ n = 0, \pm 1, \pm 2, \ldots$.

Limit Cycles
are simple closed curves in the phase plane. They correspond to periodic solutions, even for systems where the axes are not periodic.

The Poincaré-Bendixson theorem guarantees that the only bounded solution behaviours that are possible in the phase plane are fixed points or limit cycles or approaches to these or approaches to special closed curves connecting fixed points. In particular, chaotic or aperiodic bounded behaviour is not possible in the phase plane — you need a phase space that is at least three-dimensional (and nonlinear equations too) for chaos to be possible for solutions to a system of first-order differential equations.

2.1 Trajectories near critical points

It is possible to analyse the local behaviour of solution trajectories near critical points, and determine the stability of the critical points. Also, such an analysis can help piece together the global picture of what solution behaviours are possible.

At a critical point (X, Y) of the system

$$\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned}$$

we have $f(X,Y) = g(X,Y) = 0$. Now we consider small deviations $\bar{x}$, $\bar{y}$ from equilibrium

$$x = X + \bar{x} \ , \quad y = Y + \bar{y}$$

and we use Taylor series expansions in $\bar{x}$, $\bar{y}$. Then

$$\dot{x} = \dot{X} + \dot{\bar{x}}$$

and

$$f(X+\bar{x}, Y+\bar{y}) = f(X,Y) + \left.\frac{\partial f}{\partial x}\right|_{(X,Y)} \bar{x} + \left.\frac{\partial f}{\partial y}\right|_{(X,Y)} \bar{y} + \ldots$$

We truncate at the linear terms, for small $\bar{x}$, $\bar{y}$, to obtain equations that are linearised about the fixed points

$$\dot{\bar{x}} = f_x(X,Y)\bar{x} + f_y(X,Y)\bar{y} = a\bar{x} + b\bar{y} \ \ \text{(say)} \ ,$$

$$\dot{\bar{y}} = g_x(X,Y)\bar{x} + g_y(X,Y)\bar{y} = c\bar{x} + d\bar{y} \ \ \text{(say)} \ .$$

Solutions are

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} e^{mt}$$

provided that

$$\begin{pmatrix} a-m & b \\ c & d-m \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ .$$

Nontrivial solutions require the determinant of the matrix to be zero, so that

$$m^2 + \beta m + \gamma = 0 \ , \quad \beta = -(a+d) \ , \ \ \gamma = ad - bc \ . \tag{12}$$

The stability and local behaviour of solutions near (X,Y) depends on the solutions m to this characteristic equation, which are simply the eigenvalues of the Jacobean matrix. Cases to consider are:

(I)
If both roots of eqn (12) are real, then it is a *saddle point* if one root is positive and one root is negative, or a *stable node* (or a *sink*) if both roots are negative, or an *unstable node* (or a *source*) if both roots are positive.

(II)
If the roots of eqn (12) are not real, they must form a complex conjugate

pair. The solutions then oscillate or spiral near the fixed point. If m has a negative real part, solutions form *stable spirals* (also called a sink) and approach the fixed point as time increases. If m has a positive real part, solutions form *unstable spirals* (also called a source) and leave the fixed point as time increases. A zero real part is a special case, with orbits determined by the next term in the Taylor series expansion, called a *center*, or *neutrally stable*.

A diagram that summarises all of these results is shown below. It is the stability diagram for the phase plane, and shows solution behaviour in the parameter space (γ, β).

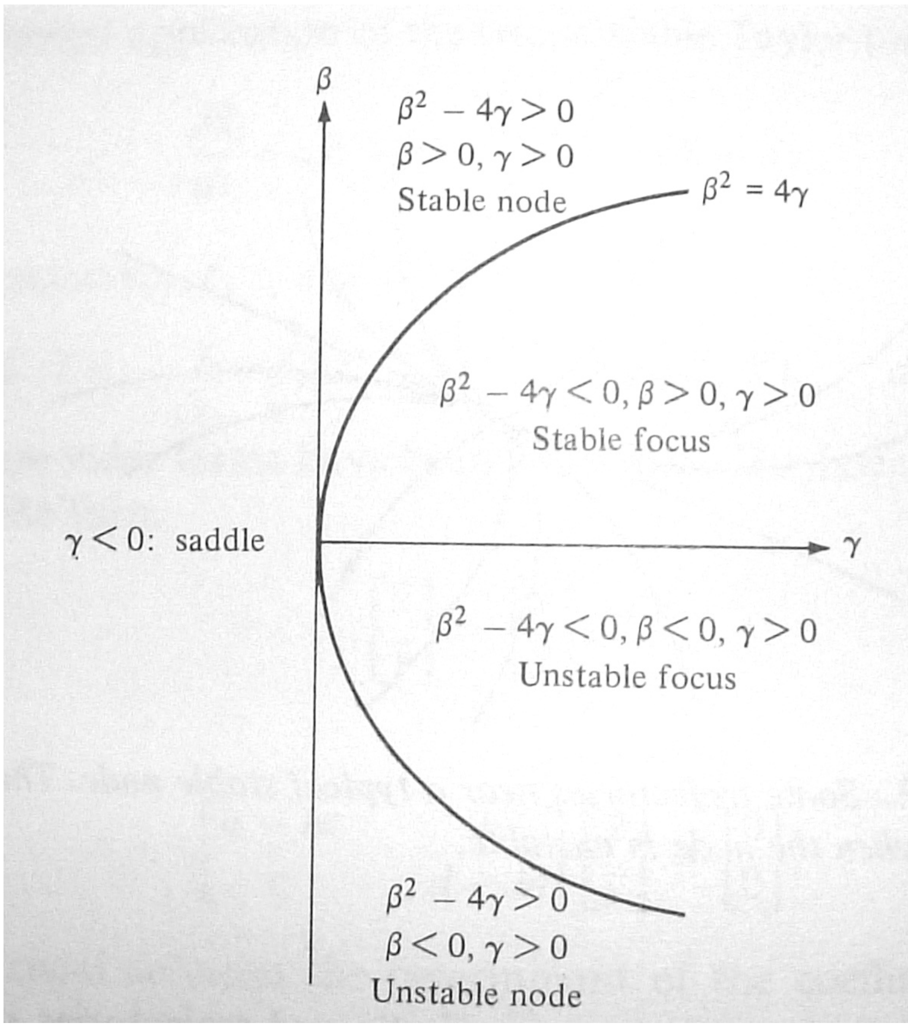


Figure 5: The stability diagram for a general phase plane

Example - rabbits and sheep

This is an example of the classic *Lotka-Volterra* model of competition between two species. Let's consider rabbits ($x(t)$ of them) and sheep ($y(t)$ of them), competing for the same grass with a limited supply. We ignore all other effects, like predators, seasonal effects, other food. Two main effects are modelled:

1. Each species would grow to a certain size, called its *carrying capacity*, in the absence of the other species. We assume a net growth rate r (births minus deaths) that is exponential, so that for example $\dot{x} = rx$, and that above a certain size (the carrying capacity K) the growth rate becomes negative (death rate higher than birth rate). A model equation that does this is the *logistic equation*
$$\dot{x} = rx\left(1 - \frac{x}{K}\right)$$
first suggested by Verhulst in 1838.

2. When rabbits and sheep meet, there is trouble over who gets to eat that patch of grass. We assume the number of encounters is proportional to both populations, and that encounters reduce both growth rates.

A model that incorporates these assumptions is

$$\begin{aligned} \dot{x} &= x(3 - x - 2y) \\ \dot{y} &= y(2 - x - y)\ . \end{aligned}$$

Fixed points are $(0,0)$, $(0,2)$, $(3,0)$, and $(1,1)$. Stability requires we find the eigenvalues λ of the Jacobean

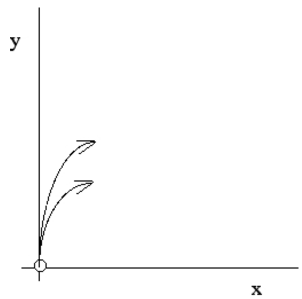
$$\begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix} .$$

At $(0,0)$ the Jacobean is

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

and eigenvalues are 3, 2. Hence $(0,0)$ is an unstable node, or a source. Trajectories leave origin parallel to the smallest (slowest) eigenvector for

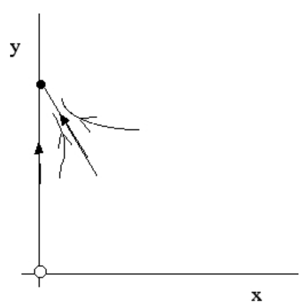
$\lambda = 2$, that is, tangent to the y-axis, as illustrated in the phase portrait:



At $(0, 2)$, the Jacobean is

$$\begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$$

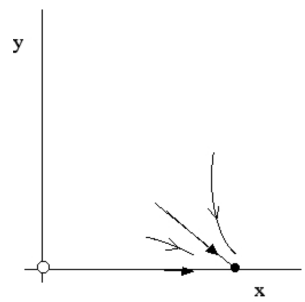
and eigenvalues are -1, -2, and the fixed point is a sink or stable node. Trajectories approach along the eigenvector corresponding to the eigenvalue -1, direction $(1, -2)$, as illustrated in the phase portrait:



At $(3, 0)$, the Jacobean is

$$\begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$$

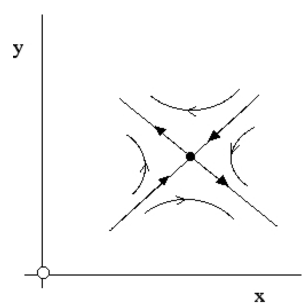
and eigenvalues are -3, -1, and the fixed point is a sink or stable node. Trajectories approach along the eigenvector corresponding to the eigenvalue -1, direction $(3, -1)$, as illustrated in the phase portrait:



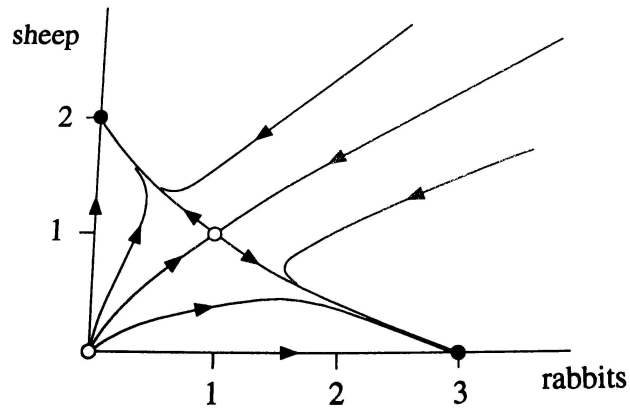
At $(1, 1)$, the Jacobean is

$$\begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

and eigenvalues are $-1 \pm \sqrt{2}$, and the fixed point is a *saddle point.* Trajectories approach along the eigenvector corresponding to the eigenvalue $-1-\sqrt{2}$, and depart along the eigenvector corresponding to the eigenvalue $-1+\sqrt{2}$, as illustrated in the phase portrait:



These local pictures of phase space behaviour may be combined to get a global view of solution trajectories. Also note that when $x = 0$, then $\dot{x} = 0$, so the y-axis is a trajectory. Similarly, when $y = 0$, then $\dot{y} = 0$, so the x-axis is a trajectory. See the phase portrait below for a computer-generated plot of the key trajectories in the first quadrant.



Note that most initial conditions lead to one of the species dominating, and the other species becoming extinct. The curve separating the first quadrant into two regions (*basins of attraction* of the two stable fixed points), one in which rabbits become extinct and one in which sheep become extinct, is also the *stable manifold* of the saddle point.

2.2 Energy

It is also useful to consider conservation of energy, when trying to understand solution behaviour. We use the nonlinear pendulum as an example again.

Recall the rescaled equation for the nonlinear pendulum,

$$\ddot{\theta} + \sin\theta = 0 \ , \quad \theta(0) = a \ , \ \dot{\theta}(0) = b \equiv \Omega/\omega_0 \ , \tag{13}$$

where time has been rescaled using $\omega_0 = \sqrt{g/L}$, and θ is the angle of the pendulum in radians (which is dimensionless).

Earlier, we noted that conservation of energy was revealed by multiplying eqn (13) through by $\dot{\theta}$, so that

$$\frac{d}{dt}\left(\frac{\dot{\theta}^2}{2} - \cos\theta\right) = 0 \ . \tag{14}$$

This equation implicitly defines constant energy curves in the phase plane, which solutions must lie on. The initial conditions determine the energy or curve which is traced. Rewriting eqn (14) in the dimensional form

$$\frac{d}{dt}\left[\frac{1}{2}m(L\dot{\theta})^2 + mgL(1-\cos\theta)\right] = 0 \tag{15}$$

expresses conservation of energy for the nonlinear pendulum, with the first term being kinetic energy ($\frac{1}{2}mv^2$ where v is linear velocity), and the second term being potential energy mgh (you can check the geometry to see that $L(1-\cos\theta)$ is the height h of the pendulum mass above its lowest point).

Damped Nonlinear Pendulum

An modification of the nonlinear pendulum equation that includes a term modelling a damping force (which is proportional to velocity in magnitude and is directed in the opposite direction to motion) is

$$\ddot{\theta} + \nu\dot{\theta} + \sin\theta = 0\ , \tag{16}$$

where $\nu > 0$ is a measure of friction, perhaps at the pivot point, and is zero if there is no friction.

This may be multiplied by $\dot{\theta}$ and integrated as before, to obtain the energy equation

$$\frac{d}{dt}\left[\frac{1}{2}m(L\dot{\theta})^2 + mgL(1-\cos\theta)\right] = -mL\nu\dot{\theta}^2\ . \tag{17}$$

Note that for $\nu > 0$, the rate of change of energy (given by the right-hand side of the above equation) is then negative, so that energy decreases for the damped pendulum. Note that the rate of energy damping varies with velocity, as expected.

The Phase Plane again

A useful way to view the phase portrait for the undamped nonlinear pendulum is to explicitly acknowledge the periodicity in θ by wrapping the plane

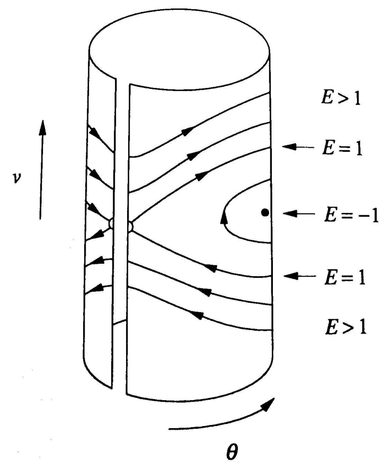


Figure 6: An alternative view of the phase plane for the nonlinear pendulum, recognising the periodicity in the angle variable.

into a cylinder with θ varying in the angular direction, and the angular velocity $\dot{\theta}$ still being the vertical coordinate, as in fig (6).

A further useful view for understanding different energy levels, noting the symmetry about the θ axis, is to bend the tube in fig (6) in two (or into a "U" shape) so that energy is now the vertical axis, as in fig (7).

The phase plane for the **damped** nonlinear pendulum is illustrated in fig (8). Note that the fixed points that were centers for the undamped pendulum have now become stable spiral points, due to energy decay. The basin of attraction of the stable spiral point at origin is shaded grey. All initial conditions, except for very special ones, approach the position where the pendulum is hanging downwards. The special initial condition, on the stable manifolds of saddle points, approach (and never reach) a final position where the pendulum is upside-down.

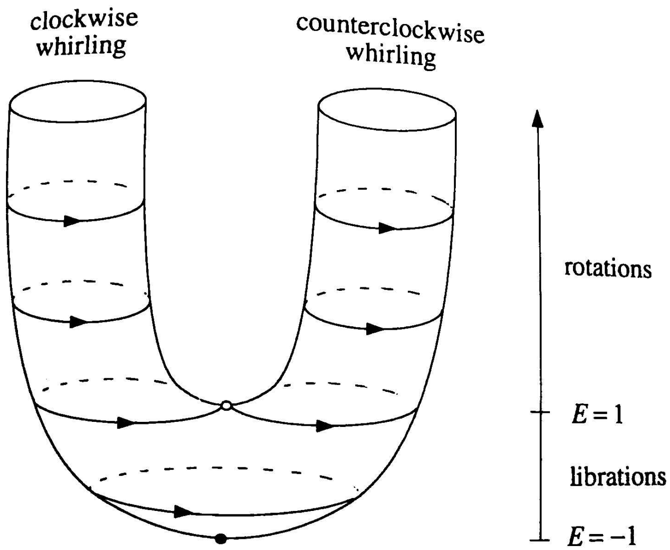


Figure 7: A view of the phase plane for the nonlinear pendulum, recognising the role played by energy conservation.

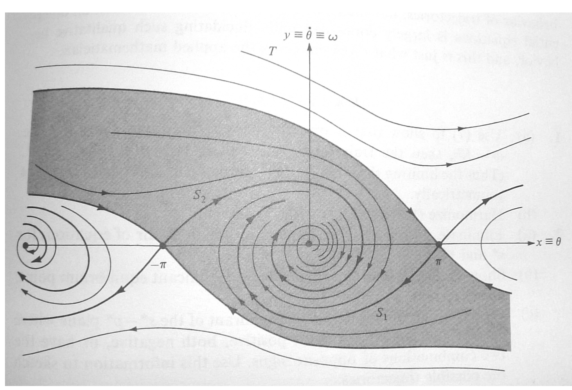


Figure 8: The phase plane for the damped nonlinear pendulum.

It is also useful to view the phase trajectories with energy plotted vertically, and θ explicitly treated as periodic as before, for the damped nonlinear pendulum, as in fig (9).

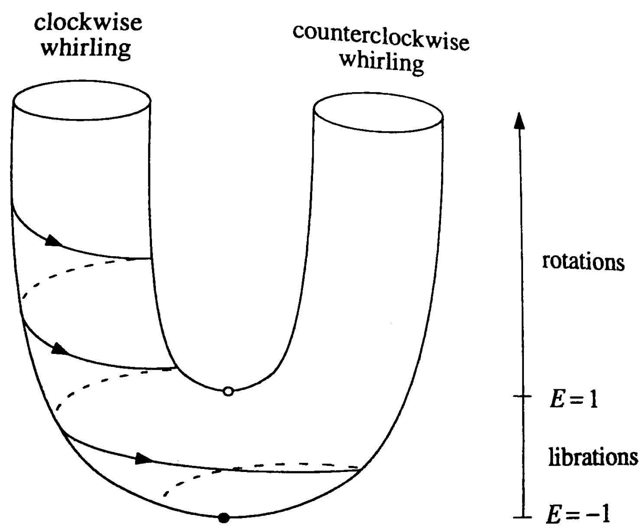


Figure 9: The phase plane for the damped nonlinear pendulum, modified to have energy on the vertical axis, and with the angle axis wrapped around to explicitly show periodicity.