School Of Mathematics, Statistics, and Operations Research
Te Kura Mātai Tatauranga, Rangahau Pūnaha

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Math 321/322/323:

Applied Mathematics

Supplementary Notes

— Special Relativity module —

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Warning:
These notes are provided as a *supplement* to the textbook.

This is a reading course, so the textbook, these notes, and various web resources should be your primary source of information.

There are still a few rough edges:

If you find errors, typos, and/or obscurities, please let me know.

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Chapter 1

Einstein's special relativity

The special theory of relativity is the theory for which Albert Einstein is most famous in the public mind. To a physicist, the special relativity is merely *one* of Einstein's many contributions to physics.

(Think: Brownian motion, photoelectric effect, general relativity, etc.)

1.1 Introduction

As almost everyone knows by now, the special relativity deals with effects that come into play at high speeds, at speeds comparable to the speed of light.

There are not that many ways in which *classical* special relativity impinges on present day technology — it affects issues of high-precision time-keeping (I'll return to this point later), the internal dynamics of the now obsolete high-voltage thermionic valves (vacuum tubes), and the kinematics of particle accelerators (colloquially called "atom smashers"). Special relativity also impinges on our technology in the annoying delay you will sometimes hear on long-distance satellite-based phone lines. Communications satellites orbit the Earth in the Clarke belt, out at 23,000 miles. For an unlucky choice of destination (and bad configuration of satellites) your phone call can be riding the radio waves for 90,000 miles or so. Even at the speed of light (186,000 miles/second) this can easily lead to an approximate half-second delay between speaking and being heard at the other end. (This problem was even more apparent on the Apollo space missions. The distance from the Earth to the Moon being about 255,000 miles, you get a $1\frac{1}{2}$ second delay each way, so at best you need to wait about 3 seconds for a response. For interplanetary probes deep in the solar system the one-way delays are often measured in hours.)

1. Special relativity is the relevant generalization of Newtonian mechanics to situations

where velocities are an appreciable fraction of the speed of light.

2. You do not want to look too closely at individual atoms.

3. Gravitational fields have to be negligible. (In fact they are idealized to zero.)

1.2 SR is a Superb theory — "Never to be Discarded"

Special relativity has been very well-tested, both experimentally and mathematically, it is definitely **Superb** in the sense of Roger Penrose's classification of physical theories. Special relativity's mathematical structure has been investigated in elaborate detail (sometimes in baroque detail), and its experimental consequences have been extensively checked (in the parameter ranges where we expect this theory to be valid, *and where we have appropriate technology*).

It is critically important to realize that in a certain sense special relativity will *never* be discarded—it's simply too useful in the range where we already *know* it works. At worst, this theory will be superseded by some more complicated "master theory" that must effectively reduce to special relativity in appropriate limits.

1.3 Textual analysis: A warning

Before we go any further, I feel that an important warning is in order: *Never attempt a comparative textual analysis of popular-level physics books* (including these notes) — the results are almost certain to be abject nonsense. By way of example, any serious literary study of Don Quixote will require you to learn the Spanish language — working only from an English language translation is never going to provide deep insight into the work. Similarly, popular level books on physics are inherently limited by the translation difficulties of adopting a natural language at the cost of excising the underlying mathematics. Do not take pretty pictures and verbal descriptions too seriously — they can be dangerously misleading — natural language is a subtle and shifting foundation on which to attempt to build physical theory.

For an example of the problems that can arise purely at the level of English language usage I need merely point out the confusion attendant on use of the word **"paradox"**. In English this word has two primary meanings. Either:

1. an *apparent contradiction* in a logically consistent theory; or

2. a real *logical inconsistency* in a truly inconsistent theory.

(There are also a few more archaic meanings that are not currently of relevance.) Worse, the most likely meaning has shifted over the past few decades.

Problems arise for instance, in the discussion of the famous "twin paradox" of special relativity. Einstein and his contemporaries used the word in the sense of an "apparent inconsistency" (what we might now call a "pseudo-paradox") and certainly did not claim or imply that special relativity was internally inconsistent. Unfortunately, many commentators have fixated attention on the word "paradox" and automatically assumed that the meaning of "real logical inconsistency" was intended, leading to discussions whose results are both pathetic and predictable. [At least half of the yelling and screaming surrounding the issue of the "twin paradox" in special relativity can be tracked down to not having a good dictionary on hand.] And this is just a simple ambiguity within the English language itself — this is not even a translation difficulty from mathematics to English. (To add to the confusion, don't forget that Einstein's native language was German, not English, and that his early works were written in German, not English.)

Another famous, or rather infamous, example of the troubles that can be caused by outright mis-translation between natural languages is that of the infamous Martian **"canali"**. Now "canali" is a perfectly good Italian word that has the English meaning of "channels" (naturally occurring, with no implication of human or alien intervention). Unfortunately US newspapers of the late 1800's mis-translated this into English as "canals" (implying they were constructed by someone or something). So much for the canals of Mars; they were never more than endless speculation heaped upon a dubious mis-translation (and a few highly ambiguous and noisy ground-based visual observations of some things that looked vaguely like channels). Still, John Carter and Barsoom will continue to live on in legend.

Another example of places where problems arise is in the discussion of the **"Einstein elevator"**. This is a gedanken-experiment (thought-experiment) devised by Einstein that argues for the complete equivalence between acceleration and an applied gravitational field. (This is the Einstein Equivalence Principle, one of the main principles underlying Einstein gravity, the general relativity, about which I will have more to say later.) More precisely, the Einstein elevator gedanken-experiment argues for the complete equivalence between acceleration and a *homogeneous* gravitational field.

Now all real gravitational fields are inhomogeneous, so the result of the Einstein elevator gedanken-experiment should really be phrased as: *"in any real gravitational field, if one has an elevator that is sufficiently small that inhomogeneities in the gravitational field can be safely ignored, then a person inside the elevator cannot tell the difference between gravity and acceleration"*

This is often shortened for convenience to: *"a person inside an elevator cannot tell the difference between gravity and acceleration"*. Unfortunately I have then (far too often) seen people who take this shortened version of the Einstein Equivalence Principle too literally. If you take the short version as *the one and only* definition of the Equivalence Principle, and then observe that real gravitational fields are inhomogeneous, than you can *mistakenly* conclude the existence of an internal inconsistency in general relativity. [This elementary mistake is unfortunately rather distressingly common; and then often leads to one particular subspecies of physics crackpottery.] Of course, what you have really deduced is that the shortened version of the Equivalence Principle is not quite precise enough — going to the long version of the Equivalence Principle removes the problem.

This all comes about because in the interests of clarity it is sometimes appropriate to delete some of the qualifying phrases that would otherwise make a popular description or an introductory textbook completely unwieldy and impenetrable — in the interests of getting any coherent message across I also shall occasionally have to resort to such trimming. But the reader should be warned that some simplification along these lines is inevitable — and if by determined textual analysis the reader discovers a logical paradox, the paradox is almost certainly a translation difficulty and not a part of the underlying physics. I trust that forewarned is forearmed.

1.4 Filtering out the nonsense

Because the theories and concepts that I am talking about in this course are so far beyond the pale of everyday experience, I think that it would be useful for the student if I were to provide some rules of thumb for filtering out the more extreme crackpot nonsense. (It is unfortunately a truism that nothing attracts the crackpots quite like the words "Einstein" and "relativity", it's like waving a red flag in front of a bull.) Now it is actually rather difficult to give hard and fast rules for detecting crackpot nonsense. Certainly any practitioner in the field can look at a specific document and within sixty seconds can come to a snap decision. Many of rules used in coming to such a conclusion are entirely heuristic and impossible to formalize in all generality. Fortunately however, a certain subset of the rules used by practicing physicists can be more or less formalized: these are the rules associated with the internal consistency of physical theories.

1.4.1 The two faces of physical theory

It is important to realize that physical theories have two main attributes that are logically distinct from one another. Physical theories must be *both* internally consistent, *and* an accurate reflection of experimental reality. To discuss the first aspect, consistency, a physical theory must be formalized as some well-defined mathematical structure, some set of equations and mathematical rules that interrelate various mathematical symbols in some way. If this mathematical structure is internally inconsistent then the theory has already failed without a single experiment being performed. The second aspect is the extent to which this mathematical structure represents reality. The various mathematical symbols appearing in the equations must be asserted to correspond to some in-principle-measurable experimental quantities. A successful physical theory is one that is mathematically consistent *and* that accurately predicts/explains/retrodicts a suitably large class of experimental results.

But note one very important point: the internal logical and mathematical consistency of the theory is not decided by experiment — consistency is purely an issue of mathematics and logic and can be settled once and for all without recourse to experiment. (It is extremely rare for a physical theory to become in any way well-established and then later fail some internal consistency checks, there are simply too many physicists working on problems and checking each other's results.) Experiment only comes in at the second stage — no matter how beautiful or internally consistent a physical theory is it is simply not useful unless it is an accurate description of how the real universe works. (Sometimes we add qualifying phrases — such as "this theory works well in thus and so a range of parameters, but is known not to accurately reflect nature if one goes outside this range of parameters".)

Very Important Point: It is absolutely critical to realise that there is an enormous difference between being "wrong" and being a "crackpot" — more on this later.

1.4.2 Rules based on mathematical consistency

These observations let us formalize several rock solid rules:

Rule 1 *If you run across someone who claims that the mathematical structure of special relativity is internally inconsistent, then you can safely ignore them: they are wrong.*

What I am saying with this rule is that the internal mathematical structure of special relativity is *provably* internally consistent, in exactly the same way in which Euclidean geometry and non-Euclidean geometries are provably mathematically consistent. (Of course you will still find some cranks who don't even believe in Euclidean geometry.) Physics textbooks explaining special relativity do not harp on this point because it is felt by most physicists to be trivial. For instance, one need merely observe that all the complications of Lorentz transformations of space and time are simply examples of a particular type of matrix multiplication on a four-dimensional vector space. The mathematical structure of special relativity is simply a special case of the mathematical structure of vector spaces and one can simply appeal to standard mathematical theorems on the internal consistency of vector space algebra. If you want to get a little more formal, you can set up a set of formal axioms for special relativity that describe it as a special type of non-Euclidean Geometry (Minkowski geometry, for example, see Reichenbach's book *"Axiomatic Relativity?"* for details). The internal consistency of special relativity is then provable in exactly the same way as the internal consistency of Euclidean geometry is provable. Remember that most physicists, and even most textbooks, consider these comments on internal consistency of special relativity to be so trivial that they are often not explicitly mentioned.

(Minkowski geometry is relatively simple, you can certainly teach it to university undergraduates, which is what this course is all about, and can even make a good stab at it with motivated high-school students. Most good university Mathematics libraries will have two or three books on axiomatic formulations of relativity.)

The flip side of Rule 1 is much more subtle:

Rule 2 *If you run across someone who does not dispute the internal consistency of special relativity, but who however claims that special relativity does not accurately reflect empirical reality then you should not necessarily reject the claim out of hand — you will have to do a little more work to judge the reasonableness of the claim.*

A claim of this type is equivalent to claiming to have experimental disproof of the applicability of special relativity to the real world. To judge such a claim requires more than mathematical manipulations — it requires you to be conversant with the current body of experimental evidence, so as to be able to see how the new experiment relates to previous experiments, and thereby make a "physics judgment" as to how reasonable and plausible the new claims are. (If the new experiment flatly contradicts ten thousand older experiments performed under similar conditions one might reasonably infer some form of "operator error".) Given the large body of empirical experimental evidence supporting the applicability of special relativity to the real world it will take stunning new experimental evidence that is truly "beyond any reasonable doubt" before any challenge to special relativity is taken seriously.

A more prosaic example is useful to get the general idea across: In days of yore, cartographers had the entertaining habit of scrawling "here be dragons" at the edges of explored territory. As more of the world was accurately mapped the putative dragon habitat shrank — to zero. If a modern cartographer were to scrawl *"here be dragons"* in the middle of Central Park, Manhattan, he or she will not be taken seriously (absent *really* compelling evidence).

The same comment applies to anyone who wants, for whatever reason, to challenge special relativity. You should find out what current experimental limits are and plan your experiments accordingly. A peculiar experimental result in the middle of a parameter region that has been well-explored by many other techniques is like a claim of a dragon sighting in Central Park — other more plausible interpretations leap readily to mind.

Rule 3 *If you run across someone who: (1) realizes that special relativity is mathematically consistent, and (2) has a well-thought out experiment that claims to demonstrate discrepancies between special relativity and the real world, and (3) has a good analysis of* *how these new experimental results match up with previous experimental results, and a good analysis of why the claimed effect does not show up in previous experiments, then (and only then) is it time to really start taking the claim seriously.*

An added bonus at this stage would be to have a carefully thought out provably-internally-consistent alternative to special relativity that reproduces special relativity in all old experiments and is in agreement with the new experiment.

I have made all of these comments specifically about Einstein's special relativity because it is the theory most well-known to the public at large, but exactly the same comments could be applied to Einstein's general relativity, and [modulo some technical quibbles], similar comments apply to quantum mechanics and quantum field theory. For instance

Rule 4 *If you run across someone who claims that the mathematical structure of general relativity is internally inconsistent, then you can safely ignore them: they are wrong.*

The point here is that there is an entire branch of mathematics (pseudo–Riemannian geometry, *aka* Lorentzian geometry, which is itself a sub-branch of differential geometry), that guarantees mathematical consistency for Einstein gravity (and many alternative theories of gravity that are sufficiently close to Einstein gravity in that they do not do too much violence to the geometric aspects of the theory).

(Lorentzian geometry is nowhere near as easy as Minkowski geometry. It is advanced undergraduate or graduate-level mathematics, though in the US it is most often taught in physics departments. In the UK, and educational systems derived therefrom, it is most often taught in mathematics or applied mathematics departments. Lorentzian geometry is also often called pseudo-Riemannian geometry.)

1.4.3 The Rough Guide to crackpot filtering

The alert reader will have noticed that all this discussion of how to filter out potentially strange and peculiar physics I have not actually defined what a crackpot is. This is partly because there is no really generally agreed upon definition (though everyone will recognize one when they run across one). Crack-pottery is associated more with a style of argument and a style of presentation than it is with the actual content. It is important to realize that people can be wrong without being crackpots, and that crackpots can accidentally be right on some issues while still remaining crackpots — crack-pottery can be loosely characterized as:

1. an inability to mentally separate the logical structure of a physical theory from issues of experimental evidence, and

2. the inability to dispassionately assess the experimental evidence, generally coupled with overwhelming arrogance [and often, unfortunately, some form of mental disease].

A very rough-and-ready guide to crackpot detection has now been circulating in the internet for a few years. The crackpot index (see the website) was developed as a humorous attempt to summarize some of the rules of thumb derived from bitter experience in the flamewars infesting the internet newsgroup sci.physics. This internet newsgroup is so heavily infested with crackpot drivel that very few (zero?) professional physicists are willing to put up with the personal abuse that generally results from giving straightforward non-crackpot answers to honest questions from genuinely curious non-experts. I shall leave it as an exercise to the reader to obtain internet access and make their own judgments. (For that matter, I should also warn readers with internet access that if you go to any of the standard internet search engines and type in the word relativity, your hits will be about 50 percent gibbering crackpot nonsense.)

You should of course, not take the final score obtained from the crackpot index too seriously. A high crackpot index merely indicates that there *might* be a problem with the document, but there may be extenuating circumstances. Likewise a low crackpot index does not guarantee that the document is correct. Unlike the relatively rigid rules I provided earlier in this chapter, the crackpot index should be used only as a rough guide.

The key issues in avoiding a high crackpot index are:

1. Think your proposal through carefully and check it for internal consistency.
2. Make sure your proposal is compatible with current experimental data.
3. Don't *ever* try to claim that classical mechanics, special relativity, general relativity, or quantum mechanics are internally inconsistent.
4. Don't try to claim that any other presently accepted theory is internally inconsistent unless you have *very* good evidence presented in a *very* clear and convincing manner.

If any of these suggestions is violated you should be very suspicious of the author's claims.

1.5 Last Words

To wrap up this introductory chapter, permit me to summarize what you should have learned:

1. Physics theories can be quality graded (**Superb**/ **Useful**/ **Tentative**) with the **Superb** theories being so well verified by experiment that any direct attack on them is simply quixotic.

2. Quantum physics and general relativity are two of the **Superb** theories. In the rest of these notes I will describe special relativity in a little more detail, and you should then have a basic understanding of what this theory entails.

3. When judging strange and exotic claims and unfamiliar physics, try to look first for issues of internal mathematical consistency, secondly for compatibility with present experiment, and only then should you worry about the details of the "new physics".

Chapter 2

Notes on notation

- One major difference between these notes and the textbook is that I will still explicitly keep track of factors of c, the speed of light.
- A professional physicist or mathematician when working in special or general relativity will often "adopt units so that $c = 1$".
- Indeed, in almost all my research work I would do so myself — and then if necessary reintroduce various factors of c in the final answer by dimensional analysis.
- So like Taylor & Wheeler, when working professionally I would measure all times and distances in seconds (or equivalently measure all times and distances in metres).
- Where I differ from Taylor & Wheeler is over the advisability of doing so the very first time you encounter special relativity. I feel that there are pedagogical advantages to keeping c around for the time being: so for these notes time is measured in seconds and distance in either "light seconds" (a physical distance of 3×10^8 metres) or just boring old metres.
- Because of this I will distinguish the coordinate x^0 from the time t by using:

  $$x^0 = ct$$

  This has the advantage that all the coordinates, x^0, x^1 x^2 and x^3 are measured in units of distance.
- I outright *refuse* to use Minkowski's $x^4 = ict$ notation — if you have never seen this notation, please do not go out of your way to encounter it. If you have previously seen the "*ict*" notation, please try to forget it — bitter experience over the last century has convinced physicists that "*ict*" is not worth the bother for special relativity, and is dangerously misleading when it comes to general relativity.

- So in these notes I will keep the factors of c explicit. If you want the equivalent formula in Taylor & Wheeler notation, just set $c \to 1$.

Chapter 3

Notes on the “light clock”

This is the simplest and easiest way anyone knows of to derive the existence of time dilation; the slowing down of moving clocks in special relativity.

We want to build a “clock” by using two mirrors and bouncing light back and forth. Assume the mirrors are at rest (meaning you are in the rest frame of the mirrors):

```
------*-----------------------------------------------------
     /|\ |                                            /|\
      |  |                                             |
      |  |                                             L
      |  |                                             |
      | \|/                                           \|/
---------*--------------------------------------------------
```

Figure 3.1: Light clock at rest.

The distance between the mirrors is L, so light takes L/c seconds to go up and L/c seconds to come back down. Each “tick” of the clock takes $2L/c$ seconds. The period of the clock (in its own rest frame) is

$$T_0 = 2L/c.$$

Now suppose the whole clock is moving sideways with speed V. (That is, we are in a reference frame in which the clock is moving.)

```
=======================>  V
-----------------------------------------------------
         /\                                   /|\
        /  \                                   |
       /    \                                  L
      /      \                                 |
     /        \                               \|/
-----------------------------------------------------
=======================>  V
```

Figure 3.2: Light clock in motion.

The light pulse takes a total time T to cross from one mirror to the other and back. We want to calculate this total time T. In its back and forth trip the light pulse moves a total vertical distance 2L (up and down; transverse dimensions are not affected in special relativity) and a horizontal distance VT. So in this reference frame the total distance travelled is (by Pythagoras' theorem)

$$(distance) = \sqrt{(2L)^2 + (VT)^2}$$

The speed of the light pulse in this reference frame is then

$$(speed) = (distance)/(time) = \sqrt{(2L)^2 + (VT)^2}/T$$

But the central weird observed fact of relativity is that light (in vacuum) always travels at the same speed regardless of which reference frame you are in, so:

$$c = \sqrt{(2L)^2 + (VT)^2}/T$$

That is:

$$c^2T^2 = (2L)^2 + V^2T^2$$

So:

$$T^2(c^2 - V^2) = (2L)^2$$

$$T^2 = (2L/c)^2/(1 - V^2/c^2)$$

$$T = \frac{T_0}{\sqrt{1 - V^2/c^2}}$$

That is:

- In its own rest frame the light clock "ticks" every T_0 seconds.
- As seen in any frame moving with speed V with respect to the light clock's rest frame, each "tick" takes $T = T_0/\sqrt{1 - V^2/c^2}$ seconds, which is always longer than T_0 — a moving clock slows down.
- (And yes, the same will happen for *any* reliable clock, no matter how constructed.)

Chapter 4

Notes on the Lorentz transformation

What we want to do is to find how to transform time and space coordinates from one frame to another.

In reference frame X the coordinates are labelled $(x^0 = ct, x, y, z)$.

In reference frame X' the coordinates are $([x^0]' = ct', x', y', z')$.

The derivation has deliberately been made slow and tedious so you can see every little step.

Note that we repeatedly use (in steps 3, 4, and 5) the fact that light rays always travel at the same speed c, while in step 1 we use use an observer who is "at rest" in one inertial frame to calibrate one of the coefficients of the Lorentz transformation in terms of the relative velocity.

4.1 Step 0:

- Whatever the relationship between X and X' is, it must be linear: Put two identical objects end on end in one reference frame and the total length will be twice the individual lengths; and this must still be true in any other reference frame. Let a clock tick twice, then the total time is just two ticks — and even if the length of each tick is different in another reference frame it is still true that two ticks take twice as long as one tick.

  (If you want to be hyper-careful: You are at at this stage implicitly using the homogeneity of space and time; in a given reference frame all ticks of the clock are the same no matter when they occur, and all metre-sticks are the same no matter where you place them.)

- In addition, if the two systems are moving with respect to each other along the

x axis (equivalently the x' axis) then the transverse directions (y, z); (y', z') are unaffected.

(If you want to be hyper-careful: There are long technical arguments based on the principle of relativity for why the transverse directions do not transform, but most people simply take this step as "obvious". If you feel at all queasy about this, you can (for now) just take it as an extra hypotheses, follow through the mathematics below, and come back later to tidy things up.)

Combined, this means that we must have:

$$ct' = Ect + Fx$$

$$x' = Gct + Hx$$

$$y' = y$$

$$z' = z$$

I now want to derive formulas for the coefficients E, F, G, H as functions of the relative velocity between the two frames.

4.2 Step 1:

Consider a person/object who is at rest (for convenience, at the origin) in reference frame X. As a function of time his "world line" is

$$P(t) = (ct, 0, 0, 0)$$

This is a line, parameterized by the coordinate t that tells you exactly where the guy is as a function of time — he's always at the origin.

Now transform to the frame X'

$$ct' = Ect + Fx = Ect$$

$$x' = Gct + Hx = Gct$$

$$y' = y = 0$$

$$z' = z = 0$$

So the "world line" of this same person, when viewed from the X' frame is

$$P'(t) = (Ect, Gct, 0, 0)$$

In the X' frame the velocity of the chap at rest in the X frame is:

$$(velocity) = (distance)/(time) = (Gct)/(Et) = (G/E)c$$

[Remember to cancel the "c" to convert x^0 to physical time]

That is:

$$G = (v/c)E$$

where v is the velocity of the X frame as viewed from the X' frame.

So we have already derived one relationship between the four quantities E, F, G, H.

4.3 Step 2:

Consider a light ray emitted (for convenience) from the origin and travelling in the $+x$ direction. As a function of time its "world line" is

$$P(t) = (ct, ct, 0, 0)$$

This is a line, parameterized by the coordinate t, that tells you exactly where the light ray is as a function of time.

Now transform to the frame X':

$$ct' = Ect + Fx = Ect + Fct = (E + F)ct$$
$$x' = Gct + Hx = Gct + Hct = (G + H)ct$$
$$y' = y = 0$$
$$z' = z = 0$$

So the "world line" of this same light ray, when viewed from the X' frame is

$$P'(t) = ([E + F]ct, [G + H]ct, 0, 0)$$

In the X' frame the velocity of the light ray is:

$$(velocity) = (distance)/(time) = ([G + H]ct)/([E + F]t) = ([G + H]/[E + F])c$$

[Remember to cancel the "c" to convert x^0 to physical time]

But since it is a light ray, its velocity in the X' frame must also be "c"!

That is:

$$[G + H]/[E + F] = 1$$

That is:

$$[G + H] = [E + F]$$

So we have now derived two relationships between the four quantities E, F, G, H.

4.4 Step 3:

Consider a light ray emitted (for convenience) from the origin and travelling in the -x direction. As a function of time its "world line" is

$$P(t) = (ct, -ct, 0, 0)$$

This is a line, parameterized by the coordinate t, that tells you exactly where the light ray is as a function of time.

As compared to Step 2 only a few $+$ signs turn into $-$ signs; the calculation is almost identical.

Now transform to the frame X':

$$ct' = Ect + Fx = Ect - Fct = (E - F)ct$$

$$x' = Gct + Hx = Gct - Hct = (G - H)ct$$

$$y' = y = 0$$

$$z' = z = 0$$

So the "world line" of this same light ray, when viewed from the X' frame is

$$P'(t) = ([E - F]ct, [G - H]ct, 0, 0)$$

In the X' frame the velocity of the light ray is:

$$(velocity) = (distance)/(time) = ([G - H]ct)/([E - F]t) = ([G - H]/[E - F])c$$

[Remember to cancel the "c" to convert x^0 to physical time]

But since it is a light ray, and its velocity in the X frame is $-c$, its velocity in the X' frame must also be "$-c$"!

That is:

$$[G - H]/[E - F] = -1$$

That is:

$$[G - H] = -[E - F] = [F - E]$$

So we have now derived three relationships between the four quantities E, F, G, H.

4.5 Step 4:

Combine the results of Step 2 and Step 3:

2: $[G + H] = [E + F] = [F + E]$

3: $[G - H] = -[E - F] = [F - E]$

Add and subtract these equations you get

$$F = G \qquad \text{and} \qquad H = E$$

But from Step 1 we already know

$$G = (v/c)E$$

Thus

$$H = E \qquad \text{and} \qquad F = G = (v/c)E$$

and the coordinate transformations read

$$ct' = E(ct + [v/c]x)$$
$$x' = E([v/c]ct + x)$$
$$y' = y$$
$$z' = z$$

It's more common to rearrange a little and to write them as

$$ct' = E(ct + [vx/c])$$
$$x' = E(x + vt)$$
$$y' = y$$
$$z' = z$$

We still have *one* unknown coefficient E to deal with...

4.6 Step 5:

Consider a light ray emitted (for convenience) from the origin and travelling in the $+y$ direction. As a function of time its "world line" is

$$P(t) = (ct, 0, ct, 0)$$

This is a line, parameterized by the coordinate t, that tells you exactly where the light ray is as a function of time.

Now transform to the frame X':

$$ct' = Ect + Fx = Ect - F0 = Ect$$

$$x' = Gct + Hx = Gct - H0 = Gct$$

$$y' = y = ct$$

$$z' = z = 0$$

So the "world line" of this same light ray, when viewed from the X' frame is

$$P'(t) = (Ect, Gct, ct, 0)$$

In the X' frame the velocity of the light ray is:

$$(velocity) = (distance)/(time)$$

Distance is calculated by the Pythagoras theorem:

$$(distance) = \sqrt{(Gct)^2 + (ct)^2} = \sqrt{G^2+1}\; ct$$

Time is just

$$(time) = Et$$

[Remember to cancel the "c" to convert x^0 to physical time]

So the speed is

$$(speed) = \frac{\sqrt{G^2+1}\; ct}{Et} = \frac{\sqrt{G^2+1}}{E} c$$

But since it is a light ray, and its speed in the X' frame must also be "c"!

That is:

$$\sqrt{G^2+1}/E = 1$$

That is:

$$G^2 + 1 = E^2$$

But, from Step 1:

$$G = (v/c)E$$

Thus:

$$(v/c)^2 E^2 + 1 = E^2$$

$$E^2[1 - (v/c)^2] = 1$$

$$E = \frac{1}{\sqrt{1-(v/c)^2}}$$

and we are now done.

(And if you stop to think about it, this step 5 is very similar to the calculation we previously did for the light clock.)

4.7 Summary:

Collecting everything:

$$ct' = \frac{ct + [vx/c]}{\sqrt{1-(v/c)^2}}$$
$$x' = \frac{x + vt}{\sqrt{1-(v/c)^2}}$$
$$y' = y$$
$$z' = z$$

It is convenient and standard to define:

$$\beta = v/c$$

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

and then write:

$$t' = \gamma(t + [vx/c^2])$$
$$x' = \gamma(x + vt)$$
$$y' = y$$
$$z' = z$$

These are the Lorentz transformations![1]

The derivation has deliberately been made slow and tedious so you can see every little step.

[1]Some older references, sometimes including Einstein himself, use an obscure obsolete notation where $\beta = 1/\sqrt{1-v^2/c^2}$. Avoid this like the plague: the modern notation, $\beta = v/c$ and $\gamma = 1/\sqrt{1-v^2/c^2}$ is now so standard that it's damn silly to do anything else.

Chapter 5

Notes on the combination of velocities

Technically and physically it's better to call it "combination of velocities" instead of the more common (but slightly misleading) phrase "addition of velocities".

Suppose we have three people: Alice, Bob, and Chuck.

Alice is moving (in the $+x$ direction) at speed v_1 with respect to Bob.

Bob is moving (in the $+x$ direction) at speed v_2 with respect to Chuck.

How fast is Alice moving with respect to Chuck? Call this v_{12}.

In Newton's physics you have the simple result:

$$v_{12} = v_1 + v_2.$$

This is simply *not true* in Einstein's physics (except in the limit as all velocities are much less than the speed of light).

The correct Einstein expression is

$$v_{12} = \frac{v_1 + v_2}{1 + [v_1 v_2 / c^2]}.$$

We will now *prove* this central fact of special relativity....

The derivation has deliberately been made slow and tedious so you can see every little step.

5.1 Derivation:

We will now derive this using the Lorentz transformations to transform time and space coordinates from one frame to another.

In reference frame X the coordinates are labelled $(x^0 = ct, x, y, z)$.
Alice is at rest in this frame.

In reference frame X' the coordinates are labelled $([x^0]' = ct', x', y', z')$.
Bob is at rest in this frame.

In reference frame X'' the coordinates are labelled $([x^0]'' = ct'', x'', y'', z'')$.
Chuck is at rest in this frame.

Remember:

It is convenient and standard to define

$$\beta = v/c$$

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

and then write

$$ct' = \gamma(ct + [vx/c]) = \gamma(ct + \beta x)$$
$$x' = \gamma(x + vt)$$
$$y' = y$$
$$z' = z$$

These are the Lorentz transformations!

5.2 Step 1:

Alice is at rest (for convenience, at the origin) in reference frame X. As a function of time her "world line" is

$$Alice(t) = (ct, 0, 0, 0)$$

This is a line, parameterized by the coordinate t that tells you exactly where Alice is as a function of time — she's always at the origin.

Now transform to the frame X'

$$ct' = \gamma_1(ct + v_1 x/c) = \gamma_1 ct$$

$$x' = \gamma_1(x + v_1 t) = \gamma_1 v_1 t$$
$$y' = y = 0$$
$$z' = z = 0$$

So the "world line" of Alice, when viewed from the X' frame is

$$Alice'(t) = (\gamma_1 ct, \gamma_1 v_1 t, 0, 0)$$

In the X' frame the velocity of Alice, who is at rest in the X frame, is:

$$(velocity)_{\text{Alice}} = (distance)/(time) = (\gamma_1 v_1 t)/(\gamma_1 t) = v_1$$

So Alice's speed with respect to Bob (who is at rest in the X' frame) is v_1.

This should not be surprising — it's just along winded way of *checking* what should be obvious.

5.3 Step 2:

Now transform to the frame X'' (Chuck's rest frame) from the X' frame (Bob's rest frame) using:

$$ct'' = \gamma_2(ct' + v_2 x'/c)$$
$$x'' = \gamma_2(x' + v_2 t')$$
$$y'' = y'$$
$$z'' = z'$$

Remember that Bob's worldline, in the X' frame, is

$$Bob'(t') = (ct', 0, 0, 0)$$

which is just another way of saying that Bob is at rest in the X' frame. If you transform this to the X'' frame (Chuck's rest frame) you will easily see that Bob's speed with respect to Chuck is v_2. (Just copy the argument of Step 1.)

$$ct'' = \gamma_2(ct' + v_2 x'/c) = \gamma_2 ct'$$
$$x'' = \gamma_2(x' + v_2 t') = \gamma_2 v_2 t'$$
$$y'' = y' = 0$$
$$z'' = z' = 0$$

So

$$Bob''(t') = (\gamma_2 ct', \gamma_2 v_2, 0, 0)$$

In the X'' frame the velocity of Bob, who is at rest in the X' frame, is:

$$(velocity)_{\text{Bob}} = (distance)/(time) = (\gamma_2 v_2 t')/(\gamma_2 t') = v_2$$

So Bob's speed with respect to Chuck (who is at rest in the X'' frame) is v_2.

This should again be a statement of the blindingly obvious.

Key step:

Now for the nontrivial part of the calculation: Let us now transform Alice's worldline into the X'' frame. Remember that in the X' frame Alice's worldline is

$$Alice'(t) = (\gamma_1 ct, \gamma_1 v_1 t, 0, 0)$$

So

$$\begin{aligned} ct'' &= \gamma_2(ct' + v_2 x'/c) \\ &= \gamma_2(\gamma_1 ct + v_2 \gamma_1 v_1 t/c) \\ &= \gamma_1\gamma_2 ct(1 + [v_1 v_2/c^2]) \end{aligned}$$

$$\begin{aligned} x'' &= \gamma_2(x' + v_2 t') \\ &= \gamma_2(\gamma_1 v_1 t + v_2 \gamma_1 ct) \\ &= \gamma_1\gamma_2(v_1 + v_2)t \end{aligned}$$

$$y'' = y' = 0$$

$$z'' = z' = 0$$

Thus Alice's worldline, as viewed in Chuck's rest frame (X''), is

$$Alice''(t) = (\gamma_1\gamma_2 ct(1 + [v_1 v_2/c^2]), \gamma_1\gamma_2(v_1 + v_2)t, 0, 0)$$

So her speed in Chuck's rest frame is

$$v_{12} = (velocity)_{\text{Alice as observed by Chuck}} = (distance)/(time) = x''/t''$$

$$= \frac{[\gamma_1\gamma_2(v_1 + v_2)t]}{[\gamma_1\gamma_2 t(1 + [v_1 v_2/c^2])]}$$

That is

$$v_{12} = \frac{v_1 + v_2}{1 + [v_1 v_2/c^2]}$$

End of proof.

The derivation has deliberately been made slow and tedious so you can see every little step.

5.4 Comments:

1) It should now be clear why you cannot just "add" velocities in the usual sense $[v_1 + v_2]$. The central point is that v_1 is a speed measured in frame X' while v_2 is a speed measured in X''.

2) Sometimes you will see the composition law written in the form

$$v_{12} = v_1 \oplus v_2 = \frac{v_1 + v_2}{1 + [v_1 v_2 / c^2]}$$

This emphasises the fact that "composition"($\oplus$) is not simply "addition".

3) We used the fact that the two velocities were parallel to each other to simplify life by choosing the coordinate axes so that we only had to consider motion in the x direction. If the two velocities are not parallel to each other life gets *much* more complicated.

4) If *both* v_1 and v_2 are small compared to "c" then it is useful to *approximate*

$$v_{12} \approx v_1 + v_2$$

Newton's result (Galileo's result) is a useful approximation at low speeds.

5) Apparently weird stuff happens if *either* $v_1 = c$ or $v_2 = c$. Think about it a little. Convince yourself it's not so weird after all.

6) Seriously weird stuff happens if *either* $v_1 > c$ or $v_2 > c$. Think about it a little. Convince yourself that, after all, this situation is even weirder than it first appears.

7) Sometimes people like to define a "rapidity parameter":

$$\xi = \tanh^{-1}(v/c); \qquad v = c\ \tanh(\xi).$$

You should amuse yourselves by showing that in terms of this rapidity parameter

$$\xi_{12} = \xi_1 + \xi_2.$$

That is: in special relativity rapidity parameters add in the usual way (at least for collinear motion). Furthermore the composition of velocities law is simply related to a quite standard hyperbolic trig identity

$$\tanh(\xi_1 + \xi_2) = \frac{\tanh(\xi_1) + \tanh(\xi_2)}{1 + \tanh(\xi_1)\tanh(\xi_2)}.$$

You can amuse yourselves by checking this.

Specifically, prove that:

$$v_{12} = v_1 \oplus v_2 = c\ \tanh\left(\tanh^{-1}(v_1/c) + \tanh^{-1}(v_2/c)\right)$$

5.5 Non-collinear velocities

In the discussion so far the velocities have all been in the same direction (collinear). What do you think might happen if you try to combine velocities that ate not collinear? Here's a few hints for the truly dedicated...

Perpendicular: The relativistic combination of perpendicular velocities $\vec{v}_1$ and $\vec{v}_2$ is particularly elegant:

$$\vec{v}_{21} = \vec{v}_1 + \sqrt{1 - v_1^2}\ \vec{v}_2 \ , \tag{5.1}$$

$$\vec{v}_{12} = \vec{v}_2 + \sqrt{1 - v_2^2}\ \vec{v}_1 \ , \tag{5.2}$$

$$||\vec{v}_{21}|| = ||\vec{v}_{12}|| = \sqrt{v_1^2 + v_2^2 - v_1^2 v_2^2} \ , \tag{5.3}$$

$$\gamma_{12} = \gamma_1 \gamma_2 \ . \tag{5.4}$$

Note specifically that

$$\vec{v}_{12} \neq \vec{v}_{21}. \tag{5.5}$$

In fact, the angle between the two is exactly the so-called Wigner rotation angle Ω:

$$\sin\Omega = \frac{v_1\gamma_1 v_2\gamma_2}{\gamma_1\gamma_2 + 1} \ , \tag{5.6}$$

$$\cos\Omega + 1 = \frac{(\gamma_1 + 1)(\gamma_2 + 1)}{\gamma_1\gamma_2 + 1} \ . \tag{5.7}$$

General: The relativistic combination of general velocities $\vec{v}_1$ and $\vec{v}_2$:

$$\vec{v}_{21} = \frac{\vec{v}_1 + \vec{v}_{2\|1} + \sqrt{1 - v_1^2}\ \vec{v}_{2\perp 1}}{1 + \vec{v}_1 \cdot \vec{v}_2} \ , \tag{5.8}$$

$$\vec{v}_{12} = \frac{\vec{v}_2 + \vec{v}_{1\|2} + \sqrt{1 - v_2^2}\ \vec{v}_{1\perp 2}}{1 + \vec{v}_1 \cdot \vec{v}_2} \ , \tag{5.9}$$

$$||\vec{v}_{21}|| = ||\vec{v}_{12}|| = \frac{\sqrt{||\vec{v}_1 + \vec{v}_2||^2 - ||\vec{v}_1 \times \vec{v}_2||^2}}{1 + \vec{v}_1 \cdot \vec{v}_2} \ , \tag{5.10}$$

$$\gamma_{12} = \gamma_1\gamma_2(1 + \vec{v}_1 \cdot \vec{v}_2) \ . \tag{5.11}$$

The Wigner rotation angle Ω:

$$\sin\Omega = \frac{v_1\gamma_1 v_2\gamma_2(1 + \gamma_1 + \gamma_2 + \gamma_{12})}{(\gamma_1 + 1)(\gamma_2 + 1)(\gamma_{12} + 1)} \sin\theta \ , \tag{5.12}$$

$$\cos\Omega + 1 = \frac{(\gamma_{12} + \gamma_1 + \gamma_2 + 1)^2}{(\gamma_1 + 1)(\gamma_2 + 1)(\gamma_{12} + 1)} \ . \tag{5.13}$$

For a continually accelerating object, the Wigner rotation leads to Thomas precession. As seen in the lab frame:

$$\frac{d\vec{\Omega}}{dt} = \vec{v}_1 \times \vec{a} \left(\frac{\gamma_1}{1 + \gamma_1} \right) \ . \tag{5.14}$$

The Thomas precession as seen in the co-moving reference frame:

$$\frac{d\vec{\Omega}}{dt} = \vec{v}_1 \times \vec{a} \left(\frac{\gamma_1^2}{1 + \gamma_1} \right) \ . \tag{5.15}$$

Further reading: For more details on this subject, see:
"Elementary analysis of the special relativistic combination of velocities, Wigner rotation, and Thomas precession".
Kane O'Donnell, Matt Visser. e-Print: arXiv:1102.2001 [gr-qc]

Chapter 6

Notes on the twin pseudo-paradox

I will now batter the twin pseudo-paradox to death with sledgehammers — analyzing the situation in a number of different ways (which ultimately *must* all agree with each other).

(I'd rather beat one well-known pseudo-paradox to death with sledgehammers than give a superficial overview of the endless list of pseudo-paradoxes that people have come up with over the years.)

In the next section I will analyse the twin pseudo-paradox using the notion of invariant interval. You will soon be sick to death of the twin pseudo-paradox since in the subsequent section I will analyze it using the Doppler effect (as seen by an observer on Earth), and in the section after that I will analyze it using the Doppler effect (as seen by an observer on the traveller), and in the section after that I will analyze it using the Lorentz transformations.

All four analyses agree with each other, as of course they must.

6.1 Analysis using the invariant interval:

Consider a pair of twins. One remains "at rest" on Earth, while the other travels on a fast rocket ship to Alpha Centauri, turns around and eventually comes back to Earth. We want to compare the total time taken for the complete trip as measured by the stay-at-home twin and the travelling twin.

6.1.1 Step 1:

First consider the spacetime diagram:

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            |   /
            |  /
            | /
START       |/
            *
```

Figure 6.1: Spacetime diagram for twin pseudo-paradox.

Where I have carefully labelled the START, TURNAROUND, and RETURN events.

In the coordinate system attached to the Earth the START event occurs at

$$E_{\text{START}} = (ct, x) = (0, 0)$$

Setting the coordinates of this event to (0,0) is a *convenience*, not a fundamental part of the physics.

Suppose the total trip, as seen by the Earth observer, takes time T.

Then in the coordinate system attached to the Earth the RETURN event occurs at

$$E_{\text{RETURN}} = (ct, x) = (cT, 0)$$

What about the TURNAROUND event? Since we assume the speed on the outward leg is the same as that on the inward leg, the turnaround must have come at the halfway point of the journey, at

$$t = T/2$$

And, if the rocket has been moving away with speed v, the position of this turnaround event must be

$$x = vt = vT/2$$

That is: in the coordinate system attached to the Earth the TURNAROUND event occurs at

$$E_{\text{TURNAROUND}} = (ct, x) = (cT/2, vT/2)$$

6.1.2 Step 2:

As measured by the travelling twin, how much time elapses between the START and TURNAROUND events?

Use the invariant interval.

Consider now only the outward leg of the journey.

Let this unknown time lapse be $\Delta t'$, and note that from the travelling twin's point of view both START and TURNAROUND events occur at the same place (which can conveniently be taken to be $x' = 0$) — this is because the travelling twin is always "at rest" in his own free float frame. So for these two events delta $x' = 0$.

Then:

$$Invariant\ interval = (c\Delta t')^2 - (\Delta x')^2 = (c\Delta t)^2 - (\Delta x)^2.$$

$$Invariant\ interval = (c\Delta t')^2 - 0 = (cT/2)^2 - (vT/2)^2.$$

$$Invariant\ interval = (c\Delta t')^2 = (cT)^2[1 - (v/c)^2]/4.$$

That is:

$$\Delta t' = (T/2)\sqrt{1 - v^2/c^2}$$

6.1.3 Step 3:

As measured by the travelling twin, how much time elapses between the TURNAROUND and RETURN events?

Use the invariant interval.

Consider now only the inward leg of the journey. (This logic is almost an exact copy of step 2).

Let this unknown time lapse be $\Delta t''$, and note that from the travelling twin's point of view both TURNAROUND and RETURN events occur at the same place (which can conveniently be taken to be $x'' = 0$) — this is because the travelling twin is always "at rest" in his own free float frame. So for these two events $\Delta x'' = 0$.

Then:

$$Invariant\ interval = (c\Delta t'')^2 - (\Delta x'')^2 = (c\Delta t)^2 - (\Delta x)^2.$$

$$\textit{Invariant interval} = (c\Delta t'')^2 - 0 = (cT - cT/2)^2 - (0 - vT/2)^2.$$

$$\textit{Invariant interval} = (c\Delta t'')^2 - 0 = (cT/2)^2 - (vT/2)^2.$$

$$\textit{Invariant interval} = (c\Delta t'')^2 = (cT)^2[1 - (v/c)^2]/4.$$

That is:

$$\Delta t'' = (T/2)\sqrt{1 - v^2/c^2}$$

You could have guessed this by symmetry:

$$\Delta t' = \Delta t''.$$

6.1.4 Step 4:

Total time taken for the trip, as measured by the travelling twin, is:

$$\begin{aligned} T(\textit{travelling twin}) &= T(\textit{outward trip}) + T(\textit{inward trip}) \\ &= \Delta t' + \Delta t'' \\ &= (T/2)\sqrt{1 - v^2/c^2} + (T/2)\sqrt{1 - v^2/c^2} \\ &= T\sqrt{1 - v^2/c^2} \\ &= T(\textit{stay at home twin})\ \sqrt{1 - v^2/c^2} \end{aligned}$$

That is:

$$T(\textit{travelling twin}) = T(\textit{stay at home twin})\ \sqrt{1 - v^2/c^2}$$

So the travelling twin observes less time to pass than does the stay-at-home twin. When he gets back from the journey he will be physically younger than his twin brother.

Warning: The "twin paradox" is an example of a"pseudo-paradox", an *apparent* contradiction in what, when you look at it carefully, is actually a perfectly correct chain of logic.

The "twin paradox" is not a "true paradox", there is no actual logical contradiction.

English is particularly bad at making the distinction between these two concepts. Good dictionaries will at least list both possible meanings of the English word "paradox":

- (1) an actual contradiction in a superficially valid chain of logic,

  and

- (2) a superficial apparent contradiction in a perfectly valid chain of logic.

All special relativity paradoxes are of type (2), and it is safer to refer to them as "pseudo-paradoxes".

6.2 Analysis using the Doppler effect (as seen by Earth):

In this section I will analyse the twin pseudo-paradox using the Doppler effect — I will concentrate on the situation as SEEN [not OBSERVED] by the stay at home twin on Earth.

Warning: The words SEEN and OBSERVED have special technical meanings in Special Relativity — see the textbook.

(In the next section I will use the Doppler effect to analyze the situation as SEEN by the travelling twin.)

Consider a pair of twins. One remains "at rest" on Earth, while the other travels on a fast rocket ship to Alpha Centauri, turns around and eventually comes back to Earth. We want to compare the total time taken for the complete trip as measured by the stay-at-home twin and the travelling twin.

6.2.1 Step 1:

First consider the spacetime diagram:

Where I have labelled the START, TURNAROUND, and RETURN events.

In the coordinate system attached to the Earth the START event occurs at

$$E_{\text{START}} = (ct, x) = (0, 0)$$

Setting the coordinates of this event to $(0, 0)$ is a *convenience*, not a fundamental part of the physics.

```
            *
RETURN      |\
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            |   /
            |  /
            | /
START       |/
            *
```

Figure 6.2: Spacetime diagram for twin pseudo-paradox.

Suppose the total trip, as seen by the Earth observer, takes time T.

Then in the coordinate system attached to the Earth the RETURN event occurs at

$$E_{\text{RETURN}} = (ct, x) = (cT, 0)$$

What about TURNAROUND? Since we assume the speed on the outward leg is the same as that on the inward leg, the turnaround must have come at the halfway point of the journey, at

$$t = T/2$$

And, if the rocket has been moving away with speed v, the position of this turnaround event must be

$$x = vt = vT/2$$

That is: in the coordinate system attached to the Earth the TURNAROUND event occurs at

$$E_{\text{TURNAROUND}} = (ct, x) = (cT/2, vT/2)$$

6.2.2 Step 2:

When does the stay at home twin on Earth SEE the turnaround event? We know its coordinates are

$$E_{\text{TURNAROUND}} = (ct, x) = (cT/2, vT/2)$$

and that light travels at speed c, so the light from this turnaround event gets back to Earth at time

$$\begin{aligned} t(light\ arrival) &= t(light\ departure) + (distance)/c \\ &= T/2 + (vT/2)/c \\ &= [T/2](1 + v/c) \\ &= t_{out} \end{aligned}$$

That is:

$$t_{out} = [T/2](1 + v/c)$$

So the coordinates of the event "Earth SEES turnaround" are

$$E_{\text{Earth SEES turnaround}} = ([cT/2](1 + v/c), 0)$$

I emphasise again that the "turnaround" and "Earth SEES turnaround" events are quite distinct.

Note that as SEEN by the Earth, the time-lapse between turnaround and return is

$$t_{in} = t_{\text{RETURN}} - t_{\text{Earth SEES turnaround}} = T - [T/2](1 + v/c)$$

That is

$$t_{in} = [T/2](1 - v/c)$$

Consistency check: $t_{out} + t_{in} = T$.

6.2.3 Step 3:

On the outward leg of the journey the frequency of light emitted by the traveller is Doppler shifted down by a factor

$$f_{out} = f_0 \sqrt{\frac{(1 - v/c)}{(1 + v/c)}}$$

On the return leg the frequency of light is Doppler shifted up by the reciprocal factor

$$f_{in} = f_0 \sqrt{\frac{(1 + v/c)}{(1 - v/c)}}$$

So suppose out travelling twin has a laser pointed back at Earth, we want to count the total number of wave crests passing by as SEEN by the Earth. We will then relate this total number of wavecrests to the elapsed time.

6.2.4 Step 4:

On the outward leg of the journey we (on Earth) SEE a frequency

$$f_{out} = f_0 \sqrt{(1 - v/c)/(1 + v/c)}$$

and we SEE the outward leg appear to last till time

$$t_{out} = [T/2](1 + v/c).$$

So the total number of wave crests passing by during the outward leg of the journey is

$$\begin{aligned} N_{out} &= f_{out}\, t_{out} \\ &= f_0 \sqrt{(1 - v/c)/(1 + v/c)}\, [T/2](1 + v/c). \\ &= [f_0\, T/2]\, \sqrt{(1 - v/c)(1 + v/c)} \\ &= [f_0\, T/2]\, \sqrt{1 - v^2/c^2}. \end{aligned}$$

6.2.5 Step 5:

On the inward leg of the journey we (on Earth) SEE a frequency

$$f_{in} = f_0 \sqrt{(1 + v/c)/(1 - v/c)}$$

and we SEE the inward leg appear to last a time

$$t_{in} = [T/2](1 - v/c).$$

So the total number of wave crests passing by during the inward leg of the journey is

$$\begin{aligned} N_{in} &= f_{in}\, t_{in} \\ &= f_0 \sqrt{(1 + v/c)/(1 - v/c)}[T/2](1 - v/c). \\ &= [f_0\, T/2]\, \sqrt{(1 + v/c)(1 - v/c)} \\ &= [f_0\, T/2]\, \sqrt{1 - v^2/c^2} \\ &= N_{out} \end{aligned}$$

Note the symmetry

$$N_{out} = N_{in}$$

6.2.6 Step 6:

The total number of wavecrests passing the Earth during the whole trip is

$$\begin{aligned} N &= N_{out} + N_{in} \\ &= [f_0\, T/2]\, \sqrt{1 - v^2/c^2} + [f_0\, T/2]\, \sqrt{1 - v^2/c^2} \\ &= [f_0\, T]\, \sqrt{1 - v^2/c^2}. \end{aligned}$$

6.2.7 Step 7:

But the total number of wavecrests seen by the Earth equals the total number of wavecrests emitted by the travelling twin, and as far as he was concerned they were being emitted with frequency f_0. Therefore the total elapsed time as MEASURED by the travelling twin is

$$\begin{aligned} T_{traveller} &= N/f_0 \\ &= [f_0\, T]\, \sqrt{1 - v^2/c^2}/f_0. \\ &= T\, \sqrt{1 - v^2/c^2} \\ &= T_{stay-at-home}\, \sqrt{1 - v^2/c^2}. \end{aligned}$$

That is

$$T_{traveller} = T_{stay-at-home}\, \sqrt{1 - v^2/c^2}.$$

Which is (of course) exactly the same result as obtained by other methods.

6.3 Analysis using the Doppler effect (as seen by the traveller):

In this section I will analyse the twin pseudo-paradox using the Doppler effect — I will concentrate on the situation as SEEN [not OBSERVED] by the travelling twin.

(In the previous section I have used the Doppler effect to analyze the situation as SEEN by the stay at home twin on Earth.)

Consider a pair of twins. One remains "at rest" on Earth, while the other travels on a fast rocket ship to Alpha Centauri, turns around and eventually comes back to Earth. We want to compare the total time taken for the complete trip as measured by the stay-at-home twin and the travelling twin.

6.3.1 Step 1:

First consider the spacetime diagram:

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RETURN      |\
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            |     \
            |      \. TURNAROUND
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            |     /
            |    /
            |   /
            |  /
            | /
START       |/
            *
```

Figure 6.3: Spacetime diagram for twin pseudo-paradox.

Where I have labelled the START, TURNAROUND, and RETURN events.

In the coordinate system attached to the Earth the START event occurs at

$$E_{\text{START}} = (ct, x) = (0, 0)$$

Setting the coordinates of this event to $(0, 0)$ is a *convenience*, not a fundamental part of the physics.

Suppose the total trip, as seen by the Earth observer, takes time T.

Then in the coordinate system attached to the Earth the RETURN event occurs at

$$E_{\text{RETURN}} = (ct, x) = (cT, 0)$$

What about TURNAROUND? Since we assume the speed on the outward leg is the same as that on the inward leg, the turnaround must have come at the halfway point of the journey, at

$$t = T/2$$

And, if the rocket has been moving away with speed v, the position of this turnaround event must be

$$x = vt = vT/2$$

That is: in the coordinate system attached to the Earth the TURNAROUND event occurs at

$$E_{\text{TURNAROUND}} = (ct, x) = (cT/2, vT/2)$$

6.3.2 Step 2:

On the outward leg of the journey the frequency of light emitted by the Earth and seen by the traveller is Doppler shifted down by a factor

$$f_{out} = f_0 \sqrt{\frac{(1 - v/c)}{(1 + v/c)}}$$

On the return leg the frequency of light is emitted by the Earth and seen by the traveller Doppler shifted up by the reciprocal factor

$$f_{in} = f_0 \sqrt{\frac{(1 + v/c)}{(1 - v/c)}}$$

So suppose the Earth twin has a laser pointed at the travelling twin, we want to count the total number of wave crests passing by as SEEN by the travelling twin. We will then relate this total number of wavecrests to the elapsed time.

6.3.3 Step 3:

On the outward leg of the journey we (the travelling twin) SEE a frequency

$$f_{out} = f_0 \sqrt{\frac{(1 - v/c)}{(1 + v/c)}}$$

and the outward leg lasts some unknown time t_{out}, as measured by the travelling twin.

So the total number of wave crests passing by is

$$\begin{aligned} N_{out} &= f_{out}\, t_{out} \\ &= f_0 \sqrt{\frac{(1 - v/c)}{(1 + v/c)}}\, t_{out} \\ &= [f_0\, t_{out}] \sqrt{\frac{(1 - v/c)}{(1 + v/c)}}. \end{aligned}$$

6.3.4 Step 4:

On the inward leg of the journey we (the travelling twin) SEE a frequency

$$f_{in} = f_0 \sqrt{\frac{(1+v/c)}{(1-v/c)}}$$

and the inward leg lasts some unknown time t_{in}, as measured by the travelling twin.

So the total number of wave crests passing by is

$$\begin{aligned} N_{in} &= f_{in}\, t_{in} \\ &= f_0 \sqrt{\frac{(1+v/c)}{(1-v/c)}}\, t_{in} \\ &= [f_0\, t_{in}]\, \sqrt{(1+v/c)/(1-v/c)} \end{aligned}$$

6.3.5 Step 5:

Since the twin is moving just as fast on the way out as on the way back

$$t_{out} = t_{in}$$

Even if we don't know how long this time is, we do know it's the same for both legs of the journey and so

$$T_{traveller} = t_{out} + t_{in} = 2t_{out} = 2t_{in}$$

6.3.6 Step 6:

The total number of wavecrests emitted from Earth and passing the traveller during the whole trip is

$$\begin{aligned} N &= N_{out} + N_{in} \\ &= [f_0\, t_{out}]\, \sqrt{(1-v/c)/(1+v/c)} + [f_0 t_{in}]\, \sqrt{(1+v/c)/(1-v/c)} \\ &= [f_0\, t_{out}](\sqrt{(1-v/c)/(1+v/c)} + \sqrt{(1+v/c)/(1-v/c)}) \\ &= [f_0\, t_{out}]([(1-v/c)+(1-v/c)]/\sqrt{1-v^2/c^2}) \\ &= [f_0\, t_{out}](2/\sqrt{1-v^2/c^2}) \\ &= [f_0\, T_{traveller}]/\sqrt{1-v^2/c^2} \end{aligned}$$

6.3.7 Step 7:

But the total number of wavecrests seen by the traveller equals the total number of wavecrests emitted by the Earth twin, and as far as he was concerned they were being emitted with frequency f_0. Therefore the total elapsed time as MEASURED by the Earth twin is

$$\begin{aligned} T_{earth} &= N/f_0 \\ &= \left\{ [f_0 T_{traveller}]/\sqrt{1-v^2/c^2} \right\} /f_0. \\ &= T_{traveller}/\sqrt{1-v^2/c^2}. \end{aligned}$$

That is:

$$T_{traveller} = T_{earth}\sqrt{1-v^2/c^2}$$

Which is (of course) exactly the same result as obtained by other methods.

6.4 Analysis using Lorentz transformations:

By now you have seen the "twin paradox" [remember it's a "pseudo"-paradox] handled in three different ways:

(1) using the invariant interval,

(2) using the Doppler shift to analyze clock pulses as SEEN by the stay-at-home twin on Earth, and

(3) using the Doppler shift to analyze clock pulses as SEEN by the travelling twin on the rocket ship.

The point of all this repetition is to show you that the same calculation can be done many different ways, and that no matter how you do it, the same physics question always results in the same physics answer.

Now we will use the Lorentz transformations to get the same result.

(That is, concentrate on what is OBSERVED; with the usual special relativity warning that the word OBSERVE is not identical to the word SEE.)

```
            *
RETURN      |\
            | \
            |  \
            |   \
            |    \
            |     \
            |      \. TURNAROUND
            |      /
            |     /
            |    /
            |   /
            |  /
            | /
START       |/
            *
```

Figure 6.4: Spacetime diagram for twin pseudo-paradox.

6.4.1 Step 1:

Consider the spacetime diagram:

Where I have labelled the START, TURNAROUND, and RETURN events.

Suppose that, as observed by the twin left behind on Earth, the trip takes total time T, and that the travelling twin is observed to move away with velocity v on the outward leg, and return with velocity v in the inward leg of the journey.

In the coordinate system of the Earth, which we can approximate to be a free-float frame (inertial frame, free-fall frame), the START event occurs at

$$E_{\text{START}} = (ct, x) = (0, 0)$$

[Setting the coordinates of this event to $(0, 0)$ is a *convenience*, not a fundamental part of the physics.]

Then in the coordinate system attached to the Earth the RETURN event occurs at

$$E_{\text{RETURN}} = (ct, x) = (cT, 0)$$

What about TURNAROUND? Since we assume the speed on the outward leg is the same as that on the inward leg, the turnaround must have come at the halfway point of the journey, at

$$t = T/2$$

And, if the rocket has been moving away with speed v, the position of this turnaround event must be

$$x = vt = vT/2$$

That is: in the coordinate system attached to the Earth the TURNAROUND event occurs at

$$E_{\text{TURNAROUND}} = (ct, x) = (cT/2, vT/2)$$

That is the coordinates of the 3 key events are:

$$E_{\text{START}} = (ct, x) = (0, 0)$$

$$E_{\text{TURNAROUND}} = (ct, x) = (cT/2, vT/2)$$

$$E_{\text{RETURN}} = (ct, x) = (cT, 0)$$

6.4.2 Step 3:

The Lorentz transformations appropriate to the outward leg are:

$$ct' = \gamma(ct \pm vx/c)$$

$$x' = \gamma(x \pm vt)$$

$$y' = y \qquad \text{[superfluous]}$$

$$z' = z \qquad \text{[superfluous]}$$

For the time being I have assumed that we are all totally confused as to whether to pick the plus or minus sign, so I'll do both simultaneously and fix it up later. [This is a very useful little trick.]

Then:

$$E'_{\text{START}} = (\gamma(0 \pm 0), \gamma(0 \pm 0)) = (0, 0)$$

Big surprise, the origin maps into the origin...

For the turnaround event

$$E'_{\text{TURNAROUND}} = (\gamma(cT/2 \pm v[vT/2]/c), \gamma(vT/2 \pm vT/2))$$

But, we are supposed to be transforming into the rest frame of the outgoing rocket. So whatever its initial position was [it happens to be $x'(start) = 0$], it must have the same final position [$x'(turnaround) = 0$].

This means we must pick the minus sign above.

That is: the correct Lorentz transformation for the outgoing (departure leg) is

$$ct' = \gamma(ct - vx/c)$$

$$x' = \gamma(x - vt)$$

and

$$E'_{\text{TURNAROUND}} = (\gamma(cT/2 - v[vT/2]/c), \gamma(vT/2 - vT/2)) = ([cT/2]\ \gamma\ (1 - v^2/c^2), 0)$$

That is:

$$E'_{\text{TURNAROUND}} = ([cT/2]\ \sqrt{1 - v^2/c^2}, 0)$$

Interpretation: As measured by the departing traveller, the time elapsed till he turns on his rockets and comes screaming back at you is given in terms of T (the total trip time as seen by the Earth) by

$$T_{elapsed(outward)} = [T/2]\ \sqrt{1 - v^2/c^2}$$

Aside (not necessary for the calculation): If we now calculate the coordinates of the RETURN event in this outward moving coordinate system we get

$$E'_{\text{RETURN}} = (\gamma(cT - 0), \gamma(0 - vT)) = (cT\gamma, -vT\gamma)$$

While true, these coordinates are not particularly useful for anything.

6.4.3 Step 4:

Now analyze the inward leg (return leg) of the journey.

Since in the Lorentz transformations we took the minus sign for the outward leg of the trip, we must take the plus sign for the inward leg. That is:

$$ct'' = \gamma(ct + vx/c)$$

$$x'' = \gamma(x + vt)$$

$$y'' = y \qquad \text{[superfluous]}$$

$$z'' = z \qquad \text{[superfluous]}$$

Then:

$$E''_{\text{START}} = (\gamma(0+0), \gamma(0+0)) = (0,0) \qquad \text{[not a surprise]}$$

Similarly:

$$E''_{\text{TURNAROUND}} = (\gamma(cT/2 + vT/2), \gamma(vT/2 + vT/2)) = ([cT/2]\gamma(1 + v^2/c^2), \gamma vT)$$

[looks quite messy, patience, ...]

Finally:

$$E''_{\text{RETURN}} = (\gamma(cT + 0), \gamma(0 + vT)) = (cT\gamma, vT\gamma)$$

[looks quite messy, patience, ...]

Interpretation:

$$x''(turnaround) = \gamma\ vT = x''(return)$$

Good. This means that on the inward leg the rocket is not moving in its own reference frame. (This does not guarantee correctness but is a consistency check that had better work out; the more consistency checks you can build into the calculation the better.)

As measured by the returning traveller, the time elapsed between when he turns on his rockets and comes screaming back at you and his return to Earth is given in terms of T (the total trip time as seen by the Earth) by

$$T_{elapsed(inward)} = t''(return) - t''(turnaround)$$

Which we compute as

$$\begin{aligned} T_{elapsed(inward)} &= [T\ \gamma] - [[T/2]\ \gamma(1 + v^2/c^2)] \\ &= [T/2]\gamma(1 - v^2/c^2) \qquad \text{[slightly nontrivial subtraction]} \\ &= [T/2]\sqrt{1 - v^2/c^2} \\ &= T_{elapsed(outward)} \end{aligned}$$

Note the symmetry between the outward leg and the inward leg.

6.4.4 Step 5:

As measured by the twin on the rocket

$$\begin{aligned} T_{elapsed(rocket)} &= T_{elapsed(outward)} + T_{elapsed(inward)} \\ &= [T/2]\ \sqrt{1 - v^2/c^2} + [T/2]\ \sqrt{1 - v^2/c^2} \\ &= T\ \sqrt{1 - v^2/c^2} \end{aligned}$$

As measured by the twin on the Earth

$$T_{elapsed(earth)} = T$$

Note that these are elapsed times along two separate distinct paths through spacetime.

That is:

$$T_{rocket} = T_{earth}\ \sqrt{1 - v^2/c^2}.$$

This is (naturally), the *same answer* as obtained by the previous 3 calculations.

6.5 Summary:

We have now seen *four* different ways of battering the twin pseudo-paradox to death — all calculations agree that the "travelling twin" ages slower.

The key point this that the travelling twin follows a kinked path through spacetime. This is *qualitatively different* from the worldline of the stay at home twin who follows a straight line through spacetime. Bent paths connecting two events simply have different lengths than straight paths connecting the same to events.

In Euclidean geometry, bent paths are always longer than straight paths. In the Lorentzian geometry appropriate to special relativity it turns out that bent world lines (timelike curves) are always shorter than straight world lines (timelike curves).

Ultimately this is due to the fact that the invariant interval [which is the (3+1) dimensional analogue of the Euclidean Pythagoras theorem], contains a minus sign in addition to the 3 plus signs...

$$\Gamma(\Delta X) = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

(And no, this comment is not supposed to be obvious, think about it a little...)

Chapter 7

Notes on the "warp drive"

Anyone who has been anywhere near a television set or movie theater sometime over the past 20 years has at least heard of the concept of "warp drive" and FTL (faster-than-light) travel.

Of course the mass media does not exactly capture the full flavour of some of the high weirdness that warp drive would imply...

There are fundamental physics reasons why the scientific community is deeply suspicious of the warp drive.

Below I will walk you through one of the simpler problems associated with warp drive physics. When you dig deeper into things, life gets even messier...

What's wrong with warp drive?

This discussion will come under the heading of: "honest-to-god seriously inconsistent logical paradox".

7.1 Step 1:

Suppose we make a pact with the devil and get hold of a warp drive, that can make a spaceship travel at 1000 times the speed of light.

This is warp factor 10 in Star Trek language. According to Wikipedia

$$(speed) = (warp\ factor)^3 \times c$$

at least in the old Star trek series...

(Some science fiction fans, "trekkies", really have *way* too much time on their hands. If you think that trying to set up a precise technical definition of "warp factor" is a little "over the top", think about the mindset required to learn to read and write the Klingon language.)

We sit here on Earth and send the Enterprise out to the nearest Star, Alpha Centauri.

Give the DEPARTURE event the coordinates

$$E_{\text{DEPARTURE}} = (ct, x) = (0, 0).$$

This is simply a *convenience*, not a fundamental part of the physics...

Alpha Centauri is about 4.5 light years away.

7.2 Step 2:

As measured by someone on Earth it takes the Enterprise (4.5/1000) years [that is, 4.5×10^{-3} years] to get there.

That's 1.6 days, or 39.4 hours.

The (Earth-based) coordinates of the ARRIVAL event are

$$E_{\text{ARRIVAL}} = (ct, x) = (c \times 39.4 \text{ hours}, 4.5 \text{ light years}).$$

To cut down on the clutter, I'll call the distance to Alpha Centauri L for the rest of the problem; I won't put in numbers for L unless and until they are needed; like right at the end of the problem.) Then

$$E_{\text{ARRIVAL}} = (ct, x) = (L/1000, L)$$

7.3 Step 3:

Calculate $\gamma = (1 - v^2/c^2)^{-1/2}$ for the segment of the trip made under warp drive. That is

$$\gamma = \frac{1}{\sqrt{1 - (1000)^2}} = \frac{1}{\sqrt{-999,999}} = \frac{i}{\sqrt{999,999}} \approx \frac{i}{1000}$$

So the γ factor that comes into Lorentz contraction and time dilation is complex, in fact it's pure imaginary. This does at the very least suggest that we better think *very* carefully about what is going on.

7.4 Step 4:

Now draw a spacetime diagram, and label the events as we go.

```
            |          / (light ray)         * ARRIVAL
            |         /                   *
            |        /                 *
            |       /               *
            |      /             *
            |     /           *
            |    /         *
            |   /       *
            |  /     *
            | /   *
            |/ *
DEPARTURE   *
```

Figure 7.1: Spacetime diagram for outward leg of the warp trip (not to scale).

Q: With respect to the DEPARTURE event, is this ARRIVAL event SPACELIKE separated, TIMELIKE separated, or NULL separated (LIGHTLIKE separated)?

[See the textbook for discussion and definition of these terms.]

A: The two events DEPARTURE and ARRIVAL are SPACELIKE separated
after all the very definition of FTL implies you will be outside the light cone...

7.5 Step 5:

Q: With respect to the DEPARTURE event, is this ARRIVAL event in the ABSOLUTE FUTURE, on the FUTURE LIGHT CONE, in the AMBIGUOUS ELSEWHEN, on the PAST LIGHT CONE, or in the ABSOLUTE PAST?

[Notation: See page 182 of the textbook; what I and most other people call the ABSOLUTE FUTURE is what Taylor–Wheeler call the "active future". What I and and most other people call the ABSOLUTE PAST is what Taylor–Wheeler call the "passive past". Lastly, what I am calling the AMBIGUOUS ELSEWHEN is known by many different names: "elsewhen", the "ambiguous region", the "relative when", the "ambiguous present" — Taylor–Wheeler call it the "neutral region" or the "unreachable region".]

A: The ARRIVAL event is in the AMBIGUOUS ELSEWHEN with respect to the departure event.

(HINT: Can you already see a potential problem developing?)

7.6 Step 6:

After the Enterprise gets to Alpha Centauri and drops out of warp, (and we assume this means that it comes to rest with respect to the Earth), Captain Picard engages the impulse drive and quickly accelerates the Enterprise to 900 km/sec relative to the Earth (and away from the Earth).

Remember:

$$c = \text{(speed of light)} = 3 \times 10^8 \text{ metres/sec} = 3 \times 10^5 \text{ km/sec}$$

Evaluate

$$\beta = v/c = \frac{900\ km/sec}{3 \times 10^5\ km/sec} = \frac{3}{1000} = 3 \times 10^{-3}.$$

Evaluate $\gamma = 1/\sqrt{1-\beta^2}$; (now using the speed generated by the impulse engines).

$$\gamma = \frac{1}{\sqrt{1-(3\times 10^{-3})^2}} = \frac{1}{\sqrt{1-9\times 10^{-6}}}$$

Your calculator will probably round γ too much, so it's better to use an approximation based on the binomial expansion: $(1+x)^n \approx 1 + nx + ...$

$$\gamma = \frac{1}{\sqrt{1-9\times 10^{-6}}} = (1-9\times 10^{-6})^{-1/2} \approx 1 + 4.5 \times 10^{-6}.$$

7.7 Step 7:

Assuming the impulse drive was on for only a very short time, (that is, neglect the time taken to turn on the impulse engines and build up impulse speed), what are the coordinates of the ARRIVAL event in the reference frame of the now moving Enterprise?

(That is, do a Lorentz transformation.)

Start from the fact that

$$E_{\text{ARRIVAL}} = (ct, x) = (L/1000, L),$$

and use

$$ct' = \gamma(ct - vx/c) \approx (1) \times [L/1000 - 3 \times 10^{-3} L] = -2 \times 10^{-3} L,$$
$$x' = \gamma(x - vt) \approx (1) \times [L - (3 \times 10^{-3})(L/1000)] \approx L.$$

That is (keeping only the most significant pieces, the bits we are neglecting are always about a million times smaller than the bits we are keeping)

$$E'_{\text{ARRIVAL}} = (ct', x') \approx (-2L/1000, L).$$

Note: You cold always amuse yourself by not making any approximation here, and keeping the full exact result. (And no, this would not make any significant difference to our final conclusions.)

What are the coordinates of the DEPARTURE event in this same reference frame?

This is trivial, the Lorentz transformations are linear without an offset so $(0, 0)$ maps to $(0, 0)$. That is

$$E'_{\text{DEPARTURE}} = (ct', x') = (0, 0).$$

Note one particular act of weirdness that drops out of the analysis: as OBSERVED in the new reference frame (that of the Enterprise after switching off its impulse power), the ARRIVAL (at Alpha Centauri) event occurs earlier than the DEPARTURE (from Earth) event — the fact that the order of these events can be interchanged via an ordinary slower-than-light Lorentz transformation (that is, by using impulse engines in trekkie-speak) is a reflection of the fact that the ARRIVAL and DEPARTURE events are spacelike separated, so that with respect to each other the events are in the AMBIGUOUS ELSEWHEN.

Note: I carefully chose warp speed and impulse speed to get this flip in the time order. You might want to amuse yourself by deriving the inequality involving v_{warp} and v_{impulse} that will guarantee a flip in time ordering... (It's not that difficult to derive the inequality, and once you see the answer it should be "obvious".)

In this same reference frame, what is the distance to Earth? (As measured by the Enterprise at the instant the impulse drive was switched off.)

Well the γ factor that goes into the relevant Lorentz contraction is $\gamma \approx 1 + 4.5 \times 10^{-6}$, so the distance back to Earth is still L (near as makes no difference).

7.8 Step 8:

Now the Enterprise goes back into warp, and heads back toward the Earth at 1000 times the speed of light *with respect to the moving reference frame it was in when the warp drive* *was turned on.*

[After all, if special relativity is in any sense correct, then top speed of the warp drive has to be defined with respect to whatever frame the spaceship is in when the warp drive is turned on; if maximum warp speed is defined with respect to something else (e.g. the fixed stars), then you have done some pretty serious *additional* mutilation to special relativity over and above assuming faster-than-light (FTL) travel.]

Now roughly how long does it take to get back to earth? (As measured in the reference frame the Enterprise was in at the instant the impulse drive was switched off.)

Feel free to make a few approximations to make the answer look a little simpler — you can again drop *small* terms if they are negligible, but make sure to keep the *big* pieces.

[These approximations are not a matter of deep principle, it's just to simplify the linear algebra for you a little.]

Well the distance back to earth is still L (near as makes no difference), and the speed is still $1000c$, so it takes a time $L/(1000c)$ to get back. And we have already done this calculation

$$L/(1000c) \approx 1.6 \text{ days} \approx 39.4 \text{ hours}.$$

So the RETURN event on Earth takes place at

$$ct' \approx -2L/1000 + L/1000 \approx -L/1000$$

and

$$x' = 0$$

That is (after the Enterprise switches off the warp drive on its return to Earth):

$$E'_{\text{RETURN}} = (ct', x') = (-L/1000, 0).$$

7.9 Step 9:

Finally, translate this all back into the reference frame of the Earth. (Do an inverse Lorentz transformation.)

The relevant β and γ are those required to bleed of the 900 km/sec speed that was built up by using the impulse engines. That is:

$$\beta = 3 \times 10^{-3}.$$

$$\gamma \approx 1 + 4.5 \times 10^{-6}$$

What time is it (roughly) on Earth when the Enterprise gets back from its trip?

$$ct = \gamma(ct' + vx'c) \approx (1) \times ((-L/1000) + 0) \approx -L/1000.$$

Aside:

$$x = \gamma(x' + vt') \approx (1) \times (0 + \beta[-L/1000]) = -3 \times 10^{-6} L \approx 0$$

So

$$E_{\text{RETURN}} = (ct, x) \approx (-L/1000, 0).$$

Is there anything strange about your final result?

$$t_{\text{RETURN}} \approx -L/(1000c) \approx -1.6 \text{ days} \approx -39.4 \text{ hours...}$$

So the Enterprise gets back from its little trip about a day and a half before it leaves.

Consistency check:

$$x = \gamma(x' + vt') \approx (1) \times (0 + (3 \times 10^{-3}) \times (-L/1000)) \approx= -3 \times 10^{-6} L$$

In other words, the Enterprise has actually overshot the Earth by about 3 parts in a million. This is consistent with the fact that we were making approximations that consistently dropped one part in a million corrections to the leading order physics...

7.10 Step 10:

What should you conclude about the possibility of warp drive or FTL travel?

Let's put it politely: The hypothesis of warp drive, or FTL travel generally, is logically incompatible with standard special relativity.

(If you desperately want to believe in FTL, then at an absolute minimum you will have to make some serious modifications to special relativity, *above and beyond* what is required simply for FTL itself. We currently [2013] have no good experimental evidence that would drive us in such a direction, and standard special relativity, supplemented by general relativity whenever gravity is important, is the best game in town.)

— # # # —

Exercise: Generalize the discussion to arbitrary values of v_{warp}, v_{impulse}, and L. ♢

Exercise: While you are at it, make the calculations *exact* by solving all relevant algebraic equations analytically. ♢

Exercise: FTL communication is almost as bad as FTL travel — Look up the article in *Physical Review D* by Benford, Book, and Newcomb: "*The tachyonic anti-telephone*". (Google is a good place to start... And yes, this is Greg Benford the Science Fiction writer, who also happens to be a Professor of Physics at the University of Califirnia at Irvine.) ♢

Exercise: Learn some general relativity. Search the internet (Google again) to find the (thankfully small) number of serious scientific papers that have attempted to analyze the nature of warp drives in general relativity. The problems a warp drive engineer would encounter in general relativity are if anything considerably worse than the issues dealt with above. ♢

Exercise: Those of you who read some Science Fiction might have run across the authors Gregory Benford, Robert L Forward, John Cramer, Geoff Landis. Do a Google search on the combination Benford, Forward, Cramer, Landis, and the term "wormhole". Enjoy.

♢

Chapter 8

Coda

Between these notes, the textbook, and the various homework exercises, I hope you now have a good feel for at least introductory special relativity — and I hope that you'll be interested in learning more about both the special relativity and the general relativity [Einstein's theory of gravity].

Cheers
Matt Visser
26 February 2013

Appendix A

Appendix: The poor man's Schwarzschild solution

A.1 Basics:

The idea of this appendix is to provide a quick (and slightly dirty) *plausibility argument* for the Schwarzschild solution of general relativity.

It gives the basic ideas without too much fuss...

This is not in any way a *derivation* — for proper rigorous derivations see any textbook on general relativity.

I'll use the ideas of "free float frames", as discussed in Taylor and Wheeler, mix in a little Newtonian physics, and out will drop a good hunk of general relativity.

A.2 Free float frames:

Start with a mass M which has Newtonian gravitational potential

$$\Phi = -\frac{GM}{r}.$$

Take a bunch of free float frames out at infinity that are stationary, and drop them.

In the Newtonian approximation these free float frames pick up a speed

$$\vec{v} = -\sqrt{\frac{2GM}{r}}\hat{r}.$$

In the free float frames, physics looks simple, and the invariant interval is simply given by

$$\mathrm{d}s_{FF}^2 = -c^2\,\mathrm{d}t_{FF}^2 + \mathrm{d}x_{FF}^2 + \mathrm{d}y_{FF}^2 + \mathrm{d}z_{FF}^2.$$

where I want to emphasize that these are locally defined free-fall coordinates.

As emphasized in Taylor & Wheeler, these free-fall coordinates will only make sense over "small" regions of space and time.

A.3 Rigid frame:

Let's try to relate this to a rigidly defined surveyor's system of coordinates that is tied down at spatial infinity.

Call these coordinates t_{rigid}, x_{rigid}, y_{rigid}, and z_{rigid}.

Since we know the speed of the freely falling system with respect to the rigid system, and we assume velocities are small we can write an approximate Galilean transformation

$$\mathrm{d}t_{rigid} = \mathrm{d}t_{FF};$$

$$\mathrm{d}\vec{x}_{rigid} = \mathrm{d}\vec{x}_{FF} + \vec{v}\,\mathrm{d}t_{FF}.$$

Inverting

$$\mathrm{d}t_{FF} = \mathrm{d}t_{rigid};$$

$$\mathrm{d}\vec{x}_{FF} = \mathrm{d}\vec{x}_{rigid} - \vec{v}\,\mathrm{d}t_{rigid}.$$

A.4 Approximate metric:

Substituting

$$\mathrm{d}s_{rigid}^2 = -c^2\mathrm{d}t_{rigid}^2 + ||\mathrm{d}\vec{x}_{rigid} - \vec{v}\,\mathrm{d}t_{rigid}||^2$$

Expanding

$$\mathrm{d}s_{rigid}^2 = -[c^2 - v^2]\mathrm{d}t_{rigid}^2 - 2\vec{v}\cdot\mathrm{d}\vec{x}\,\mathrm{d}t_{rigid} + ||\mathrm{d}\vec{x}_{rigid}||^2.$$

Substituting

$$\mathrm{d}s_{rigid}^2 = -\left[c^2 - \frac{2GM}{r}\right]\mathrm{d}t_{rigid}^2 + 2\sqrt{\frac{2GM}{r}}\,\mathrm{d}r_{rigid}\,\mathrm{d}t_{rigid} + ||\mathrm{d}\vec{x}_{rigid}||^2.$$

This is only an approximation — Newton's gravity; Galilean coordinate transformations.

A.5 The miracle du jour:

The invariant interval

$$\mathrm{d}s^2_{rigid} \;\; = \;\; -\left[c^2 - \frac{2GM}{r}\right]\mathrm{d}t^2_{rigid} + 2\sqrt{\frac{2GM}{r}}\mathrm{d}r_{rigid}\,\mathrm{d}t_{rigid} + ||d\vec{x}_{rigid}||^2.$$

is an *exact* solution of Einstein's equations of general relativity.

It is the Schwarzschild solution in disguise.

If you don't believe me, feed it to Maple and have it calculate the Ricci tensor.

This is *one* representation of the space-time geometry of a Schwarzschild black hole, in a particular and relatively unusual coordinate system (the Painleve–Gullstrand coordinates).

There are many other coordinate systems you could use.

A.6 Schwarzschild radius:

You can see that something goes wrong at

$$\frac{2GM}{r_s} = c^2; \qquad\qquad r_S = \frac{2GM}{c^2}.$$

Reverend John Michell (1783);
Peter Simon Laplace (1799).

Check dimensions!

In Einstein's gravity the coefficient of $\mathrm{d}t^2_{rigid}$ goes to zero at the Schwarzschild radius; in Newton's gravity the escape velocity

$$v_{escape} = \sqrt{\frac{2GM}{R}}.$$

reaches the speed of light once $R = r_S$.

A.7 Comments:

- This sort of argument should work generically for weak fields.

- That it is exact for Schwarzschild seems to be an accident.
- This sort of approach has a good chance of working for arbitrary spherically symmetric geometries.
- This sort of approach definitely fails for the Kerr geometry (rotating black holes).
- This sort of approach should not be thought of as fundamental physics.

For technical details see:

- Heuristic approach to the Schwarzschild geometry
- Matt Visser
- e-Print Archive: gr-qc/0309072
- International Journal of Modern Physics D14 (2005) 2051-2068.
- December 2005 "Year of Physics" issue.

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[2] E. F. Taylor and J. A. Wheeler, Exploring black holes, (Addison Wesley Longman, San Francisco, 2000).