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MATH 321/322/323 Applied Mathematics T1 and T2 2013

Module on Special Relativity: Assignment 6

This last assignment covers chapter 9 of the textbook ("Gravity: Curved spacetime in action") — this chapter provides a quick introduction to general relativity.
The assignment itself is relatively short and straightforward.
Comments:

- If you want to know more about black holes you could take a look at another book:
  Exploring black holes, also written by Taylor and Wheeler.

- You could also sign up for the Math 464 (Differential geometry), and Math465 (General Relativity and Cosmology) combination for Math Honours or Physics Honours next year.

- In the appendix to the notes, titled "The poor man's Schwarzschild solution", I give a simple introduction to and motivation for the Schwarzschild geometry of general relativity.

- It would be silly not to make use of these notes.

- **From dark star to black hole:**

  Almost immediately after Newton formulated his theory of gravity, there was speculation about the possibility of "dark stars".

  (There are a couple of really *old* papers by Michell, who you have almost certainly never heard of, and Laplace, who you very definitely should have heard of.)

  Remember that in Newtonian gravity the escape velocity from a compact body of mass M and radius R is given by:

$$\frac{1}{2}(v_{\text{escape}})^2 = \frac{GM}{R}$$

This formula is derived simply by conservation of energy, trading off (Newtonian) kinetic energy for (Newtonian) gravitational potential energy. Now define

$$R_{\mathrm{dark\,star}} = \frac{2GM}{c^2}$$

Then if R is less than $R_{\mathrm{dark\,star}}$, we have v_{escape} greater than c, so that light cannot escape.

Of course, this logic is completely Newtonian and it is pretty much a *miracle* that *exactly* the same result holds in full-fledged general relativity (Einstein gravity), up to and including the precise numerical factor of 2.

In general relativity we just change the name, and call it the Schwarzschild radius:

$$R_{\mathrm{Schwarzschild}} = \frac{2GM}{c^2},$$

and we say that objects that are smaller than their Schwarzschild radius are to be called "black holes".

- If you want a bit of a challenge, look up the article "Heuristic approach to the Schwarzschild geometry", which is available as electronic article number gr-qc/0309072 at the website http://arXiv.org

  The formal published article is available as International Journal of Modern Physics **D14** (2005) 2051–2068.

  That article pushes these ideas a little further.

- If you want a *real* challenge, (and this is really tough), try to understand rotating black holes (Kerr black holes)...

- Now, on with the assignment itself.

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Assignment:

1. **Schwarzschild radius:** Calculate the Schwarzschild radii for the Moon, Earth, Jupiter, and Sun.

   (You will have to look up a few numbers! Specifically, the masses of the Moon, Earth, Jupiter, and Sun. You will also need to either find or already know the numerical value of Newton's constant and the speed of light in the SI [metric] system of units.)

   Then I want the answer in centimetres/ metres/ kilometres (as appropriate), and I want at least 3 significant figures.

2. **Schwarzschild radius:** Compare these Schwarzschild radii with the *actual* radii of these astronomical objects. (You will have to look up a few numbers! Specifically, the actual radii of the Moon, Earth, Jupiter, and Sun.)

   (a) By what factor would we have to decrease the *radius* of the Moon, Earth, Jupiter, and Sun to turn them into black holes?

   (b) By what factor would we have to decrease the *volume* of the Moon, Earth, Jupiter, and Sun to turn them into black holes?

   (c) Do you think this would be easy to do?

3. **Schwarzschild radius:** It is suspected that the core of our own Milky Way galaxy contains a super-massive black hole with a mass of about three million times that of our Sun. Calculate the Schwarzschild radius of this black hole in kilometres, in light-seconds, in astronomical units, and in light-years.

4. **The flat Earth approximation:**

   (This is a simpler example of the "poor man" discussion in the notes available on the website.)

   We know that in an inertial frame (free-float frame, free-fall frame) the invariant interval can be written

   $$\mathrm{d}s^2 = c^2\ \mathrm{d}t_{FF}^2 - \mathrm{d}z_{FF}^2,$$

   (and never mind the x and y directions because we will make the flat Earth approximation; x and y are taken to be horizontal and the z axis is taken to be pointing *down*).

Then if we drop an inertial frame, starting from $z = 0$, it picks up a speed

$$\frac{1}{2}v^2 = gh$$

where $g \approx 10$ metres/(sec)^2 is the usual acceleration due to gravity; and we are making a non-relativistic approximation that $v \ll c$.

Using the fact that we know how quickly a free-fall frame will move with respect to a fixed frame (solidly bolted to the Earth), use Newtonian transformations to write $\mathrm{d}t_{FF}$ and $\mathrm{d}z_{FF}$ in terms of $\mathrm{d}t_{\text{fixed}}$ and $\mathrm{d}z_{\text{fixed}}$.

Now use this to re-write the invariant interval in terms of fixed coordinates, instead of free fall coordinates — you will get a formula similar to but quite a bit simpler than the one I derived in the notes for the gravitational field of an isolated point mass.

(The main message you should take from all this is that physics in non-inertial frames tends to be a lot messier than in inertial frames.)

5. By the way, what *is* the speed of light in furlongs per fortnight?

   (I want at least 3 significant digits in your answer.)

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