School Of Mathematics, Statistics, and Operations Research
Te Kura Mātai Tatauranga, Rangahau Pūnaha

MATH 321/322/323	Applied Mathematics	T1 and T2 2013

Module on Quantum Mechanics: Assignment 1

- This first assignment will deal with the classical uncertainty relations and their relationship to the Heisenberg uncertainty relations..
- Read chapter 3 of the notes — the chapter on the Heisenberg uncertainty relations.
- Applied math level proofs are good enough — I do not demand absolute analytical rigour, but do want to see enough information so that any interested reader could fill in any missing steps.
- If you find any typos in the notes or assignment, please let me know.

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1. Prove that the Dirac delta function satisfies

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-i\omega[t-t'])\,\mathrm{d}\omega$$

or equivalently

$$\delta(\omega-\omega') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-i[\omega-\omega']t)\,\mathrm{d}t$$

Here the Dirac delta function is *defined* by

$$f(t) = \int_{-\infty}^{+\infty} f(t')\,\delta(t-t')\,\mathrm{d}t'$$

$$g(\omega) = \int_{-\infty}^{+\infty} g(\omega')\,\delta(\omega-\omega')\,\mathrm{d}\omega'$$

Hint: Use (without proof) the Fourier inverse theorem.
Hint: You might want to do a little digging on Wikipedia or Google.

2. Prove *Parseval's theorem.*

   That is, show that $s(t)$ and its Fourier transform $\tilde{s}(\omega)$ satisfy

$$\int_{-\infty}^{+\infty} |s(t)|^2 \, \mathrm{d}t = \int_{-\infty}^{+\infty} |\tilde{s}(\omega)|^2 \, \mathrm{d}\omega$$

   Hint: What is the Fourier representation of the Dirac delta function?
   Hint: You might want to do a little digging on Wikipedia or Google.

3. Show that in terms of the inner products we defined in the notes, where

$$\langle s_1, s_2 \rangle \equiv \int_{-\infty}^{+\infty} s_1^*(t) \; s_2(t) \; \mathrm{d}t,$$

$$\langle \tilde{s}_1, \tilde{s}_2 \rangle \equiv \int_{-\infty}^{+\infty} \tilde{s}_1^*(\omega) \; s_2(\omega) \; \mathrm{d}\omega,$$

   Parseval's theorem can be rephrased as:

$$\langle s, s \rangle = \langle \tilde{s}, \tilde{s} \rangle$$

   Hint: You may wish to use the representation of the Dirac delta function.
   Hint: You might want to do a little digging on Wikipedia or Google.

4. Prove that in terms of this inner product the operator

$$\partial_t : s(t) \to \partial_t s(t)$$

   is anti-Hermitian. This should take 2 or 3 lines at most.

   This is just a matter of following the definitions to their logical conclusion.

5. Prove the *shifting theorem*

$$\mathcal{F}[s(t + t_0)](\omega) = \exp(+i\omega \; t_0) \; \mathcal{F}[s(t)](\omega)$$

   This should take 2 or 3 lines at most.

   This is just a matter of following the definitions to their logical conclusion.

6. Prove the *modulation theorem*

$$\mathcal{F}[\exp(-i\omega_0\, t)\ s(t)](\omega) = \mathcal{F}[s(t)](\omega + \omega_0)$$

   This should take 2 or 3 lines at most.

   This is just a matter of following the definitions to their logical conclusion.

7. Explain in your own words why combining the shifting theorem lets us simplify life by considering, (without any loss of generality), the simpler case $\bar{t} = 0$, $\bar{\omega} = 0$.

   This should take 4 or 5 lines at most.

   This is just a matter of following the definitions to their logical conclusion.

8. Prove that the Fourier transform process is a unitary linear operator on "signal-space". Remember that by definition

$$\mathcal{F}^\dagger = (\mathcal{F}^*)^T$$

   and that you are trying to prove the equivalent of

$$\mathcal{F}^{-1} = \mathcal{F}^\dagger$$

   Hint: You will need to use the (unproved) Fourier inversion theorem together with properties of the inner product $\langle \_, \_ \rangle$. The rest is just a matter of following the definitions to their logical conclusion.

9. Prove that

$$\langle \tilde{s}, \omega^2 \tilde{s} \rangle = -\langle s, \partial_t^2 s \rangle$$

   This is just a matter of following the definitions to their logical conclusion.

10. Consequently show that

$$(\Delta\omega)^2 = \frac{\langle \tilde{s}, \omega^2 \tilde{s} \rangle}{\langle \tilde{s}, \tilde{s} \rangle} = -\frac{\langle s, \partial_t^2 s \rangle}{\langle s, s \rangle}$$

   This is just a matter of following the definitions to their logical conclusion.

- This now fills in *all* the missing details in the proof of the *classical* uncertainty theorem — convinced yet?

- For background and hints, you might want to do a little digging on Wikipedia or Google.

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