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MATH 321/322/323 Applied Mathematics T1 and T2 2013

Module on Mechanix: Assignment 3

- This third assignment will deal with the Hamiltonian description of classical mechanics.
- Read the chapters on the notes that deal with the Hamiltonian formulation.
- Applied math level proofs are good enough — I do not demand absolute analytical rigour, but do want to see enough information so that any interested reader could fill in any missing steps.
- If you find any typos in the notes or assignment, please let me know.

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1. Consider a Hamiltonian function of the form

$$H(p,x) = \frac{p^2}{2m} + V(x).$$

   (a) Write down Hamilton's equations.
   Physically interpret p, m, V for this particular system.

   (b) Using Hamilton's equations, solve for p as a function of m and $\dot{x}$.

   (c) Rewrite the quantity

$$T = \frac{p^2}{2m}$$

   as a function of m and $\dot{x}$, and hence provide a physical interpretation for this quantity.

2. Repeat the above question with a Hamiltonian of the form

$$H(\vec{p},\vec{x}) = \frac{|\vec{p}|^2}{2m} + V(\vec{x}),$$

   where $\vec{p}(t)$ and $\vec{x}(t)$ are now vector functions in 3-dimensional space.

   (Given what you have already done, this should be very easy.)

3. Now consider a Hamiltonian function of the form

$$H(\vec{p}, \vec{x}) = \frac{|\vec{p} - q\,\vec{A}(\vec{x})|^2}{2m} + q\,V(\vec{x}).$$

(This will be a little trickier than what you have seen previously, but patience, things will drop out nicely in the end.)

   (a) Write down Hamilton's equations, (all of them).
   [Remember to use the chain rule.]
   [Write the result as simply as possible as 2 vector equations.]

   (b) Use one of Hamilton's (vector) equations to explicitly solve for the momentum $\vec{p}$ as a function of m, q, $\dot{\vec{x}}$, and $\vec{A}(\vec{x})$.

   (c) Substitute thus result for the momentum $\vec{p}$ into the *other* (vector) Hamilton equation.
   You should find an explicit result for the second derivative $\ddot{\vec{x}}(t)$ of position $\vec{x}(t)$ as a function of m, q, $\dot{\vec{x}}$, and various partial derivatives of V and $\vec{A}$.
   Rearrange to make these results as simple as possible, but no simpler...
   [Remember to use the chain rule.]
   [Hint: If you are on the right track you should see various components of the vector $\vec{B} = \text{curl}\,\vec{A} = \nabla \times \vec{A}$ showing up at various stages of the calculation.]

   (d) Physically interpret the resulting ordinary differential equation.
      i. Specifically, the ODE for $\ddot{\vec{x}}$ has a special name.
      What is it called?
      ii. Specifically, the quantity $\vec{B} = \nabla \times \vec{A}$ has a special name. What is it called?
      iii. Specifically, physically interpret the quantities q, $V(\vec{x})$, and $\vec{A}(\vec{x})$ for this particular system. They all have special names.

4. Now consider a Hamiltonian of the form

$$H(\vec{p}, \vec{x}) = \sqrt{m_0^2\,c^4 + |\vec{p}|^2\,c^2} + V(\vec{x}).$$

(You should find this question "straightforward", I have broken things down into simple little steps.)

(a) Write down Hamilton's equations for this particular Hamiltonian.

Solve for $\vec{p}$ as a function of $\dot{\vec{x}}$, (and c and m_0).

Provide a physical interpretation.

[You may have to think a little when inverting $\vec{a}(\vec{p})$ to find $\vec{p}\left(\dot{\vec{x}}\right)$.

Hint: When in doubt, use symmetry.]

(b) Take the limit $|\vec{p}| \ll m_0 c$ in the Hamilton equations you have just derived.

What do you get? Physically interpret your result.

(c) Take the limit $|\vec{p}| \ll m_0 c$ directly in the Hamiltonian that I provided above. Be certain to keep the first non-trivial term in $|\vec{p}|$.

What do you get? Physically interpret your result.

(d) Again take the limit $|\vec{p}| \ll m_0 c$ directly in the Hamiltonian that I provided above, but now keep the first *two* nontrivial terms in $\vec{p}$.

What do you get? Physically interpret your result.

(e) Now take the limit $m_0 \to 0$ in the Hamiltonian equations of motion you derived in the first step above.

Do the equations of motion have a sensible limit?

Physically interpret your result.

(f) Now take the limit $m_0 \to 0$ directly in the Hamiltonian that I provided above.

Does the Hamiltonian have a sensible limit?

Physically interpret your result.

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