School of Geography, Environmental and Earth Sciences and School of Mathematics, Statistics and Computer Science

MATH 322/323
Module on Fluid Flow in Earth Systems
2014 Computer Section Notes-Part 2 and Assignment 5

Due 9AM Monday 13 October 2014

Section 2 of Maths 323 Fluid Flow Computer Section: Numerical models for 1D flow

Last week we were introduced to the concepts of numerical integration and differentiation, and we solved the 1D fluid flow problem by numerical integration using the trapezoidal rule. This time you will use the trapezoidal rule spreadsheet that you developed last time and we will also introduce the finite difference method by solving the same fluid flow problem, using a finite difference approach. This is a more general, and very powerful, approach, although for this particular case you could get better and quicker answers by the techniques covered last time. The finite difference approach starts directly from the differential equation, and solves it iteratively by taking numerical derivatives only—it doesn't do any explicit numerical integration in the fashion that we looked at last time.

As you showed last week, if the viscosity is a function of depth, and the pressure gradient *dp/dx* is a constant *P*, then equation 6-10 of Turcotte and Schubert is modified to become:

$$\frac{d(\mu(y)\frac{du}{dy})}{dy} = \frac{dp}{dx} = P \quad \text{(Eq. FF1)}$$

Where as usual, u is the velocity in the x direction, y is depth, and μ is viscosity.

Differentiating yields the following equation:

$$(\mu(y)\frac{d^2u}{dy^2}) + \frac{du}{dy}\frac{d\mu}{dy} = P \quad \text{(Eq. FF2)}$$

An extremely simple, brute-force finite difference scheme can be written by replacing all the derivatives by their numerical equivalents, as discussed in last week's notes on finite difference solutions, so that the values at the jth point become

$$\mu_j \frac{(u_{j+1} - 2u_j + u_{j-1})}{\Delta y^2} + (\frac{(u_{j+1} - u_{j-1})}{2\Delta y})(\frac{(\mu_{j+1} - \mu_{j-1})}{2\Delta y}) = P \quad \text{(Eq. FF3)}$$

The 1-D excel spreadsheet you will be given solves this problem. This spreadsheet does not include any changes in time. Every iteration of this spreadsheet simply gives a more accurate estimate of the steady state velocity distribution. The spreadsheet works as follows: If you put a "0" in the column labeled "X", it will reset all the velocities to their initial values when you hit the "Ctrl +" key. If you then put a "1" in the "X" position, it will iterate on the solution every time you hit "Ctrl +".

Assignment 5. Due 9AM Monday 13 October 2014

For each section, include plots and print out your examples, and provide a discussion of what you have learned. You may hand in spreadsheets via email as supporting material, but you will be marked mainly on your printed version (legible handwritten sections are fine).

1. In your spreadsheet for the trapezoidal rule integration (that you developed for last week's assignment), add more columns to calculate the strain rate and the stress. Show that they give the expected responses for constant viscosity for Couette flow. (Hint: Look at the first part of Ch. 6 of Turcotte and Schubert again. Couette flow is defined as ($P=0, u_0\neq 0$). What is the relation between strain rate and the velocity gradient? What is the relation between stress and strain rate? Also—let me know if you think your spreadsheet from last time was not working—I can supply a working spreadsheet) [5 marks]

2. In the trapezoidal rule integration spreadsheet, modify the viscosity structures to change to a layered viscosity structure (i.e., viscosity varies with depth in the sense that the viscosity is constant in one layer, then changes in another layer, and then again; use at least three layers), and see how the velocity, strain rate and stress change with depth. Plot the values along with the viscosity. Explain the variations (or lack thereof) in velocity, strain rate and stress that you see. [10 marks]

3. Show that the equations FF2 and FF3 follow directly from FF1. (Hint: this is just a 1st year calculus problem—nothing tricky, but last week's notes are helpful to get from FF2 to FF3.) Solve FF3 for u_j (i.e., rearrange it so that u_j is on the left, and everything else is on the right.) Show by example that the new Finite Difference Excel spreadsheet solves this equation (i.e., print out the equations for one of the cells in the middle of the table and compare it to your equations to see if they are the same). Notice that the solution means that in each new iteration, the new value of every element of velocity depends on the average of the velocities of all the points around it in the last iteration, as well as on the products of the differences in the velocity of the points around it and the differences in viscosity around it. So it is a type of "averaging" function. You will see this same behaviour when we look at 2-D finite difference schemes. [5 marks]

4. The Excel spreadsheet and the numerical integration by trapezoidal rule solve the same problem. Modify the new spreadsheet to calculate strain rate and stress. Verify that, for constant viscosities, both spreadsheets give the expected distribution of velocity, stress and strain rate for the two cases covered in Turcotte & Schubert: Couette flow ($P=0, u_0\neq 0$), and the case $u_0=0, P\neq 0$, and check that both codes also give the same response as the book. (Note that, if you put truly realistic mantle values of viscosity into the equations, you may run into problems. The relative viscosity differences between layers is all that should matter, so try using viscosities with values closer to 1 instead.) [10 marks]

5. Compare the two spreadsheets for Couette flow with a new layered viscosity structure, with 3 layers of different viscosity. Researchers have suggested that some finite difference schemes for examining changes in flow pattern with time will not work properly if the viscosity changes by more than a factor of 3 between each level. Check if this holds for your two types of spreadsheets. Which ones seem to give the most stable results for large changes in viscosity? Is velocity or strain rate or stress most stable, and why? [10 marks]

6. Modify the new spreadsheet to try linear and exponential change in viscosity with depth. How many iterations are needed for the new spreadsheet to get to a solution in each case? [10 marks]

7. Finally, apply the problem to the flow of the asthenosphere beneath a moving plate, e.g., the oceanic lithosphere. Choose appropriate values for P, u_0 and a viscosity structure appropriate to the mantle, and use the spreadsheet that you think is most able to handle the problem. However, use only relative values of viscosities rather than absolute values, since the computer doesn't like to handle big numbers. (See table 3 below for one model, modified by, e.g., setting μ=1 in the bottom layer, 10 in the next one up, 100 above that, etc. and P=0 is a good value. But note the problem discussed above in question 5 about numerical instability—so you might want to spread the changes over more layers.) If you use 800 km as the maximum depth (0 velocity) of your model, what velocities do you find at a depth of 100 km? Where do the largest strains occur? What if you change the model to have a maximum depth of 400 km? [10 marks]

8. Provide a discussion of what you have learned from this section [5 marks].

Hints: Excel spreadsheets: Some versions won't accept implicit iterations (circular references) by default. To fix it, or to change the number of iterations performed with each button push, go to Tools, Options, Calculations Check that Iterations is checked. Change the Maximum Iterations box to allow more times.

Table 3 below from Savage et al. (2004 Geol Soc. London Special Publication Savage, M. K., K. M. Fischer and *C. E. Hall*, Strain modelling, seismic anisotropy and coupling at strike-slip boundaries: Applications in New Zealand and the San Andreas Fault, in *Vertical Coupling and Decoupling in the Lithosphere*, Editors: Grocott, J., Tikoff, B., McCaffrey, K. J. W. & Taylor, G., Geological Society of London, Special Publication, 227, 9-40, 2004). See me if you want a pdf of the whole article:

Table 3. Pacific Plate viscosity for WUS+PAC model

Depth to bottom of layer (km)	Viscosity (Pa s)
40	$1x10^{25}$
45	$1x10^{24}$
50	$1x10^{23}$
55	$1x10^{22}$
60	$1x10^{21}$
70	$1x10^{20}$
800	$1x10^{19}$