Notes for Assignment 3, Maths 323 Fluids Module 2014-last time:

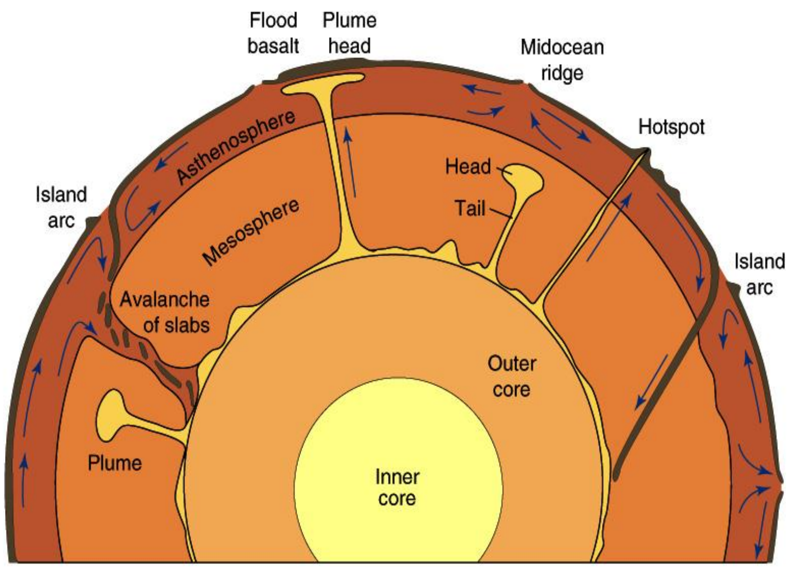
1) Concepts from Turcotte & Schubert Ch. 4 needed in Ch. 6
2) Thermal expansion and Plume Heads and Tails (Section 6-15)
3) Heat conduction equation in a moving medium:

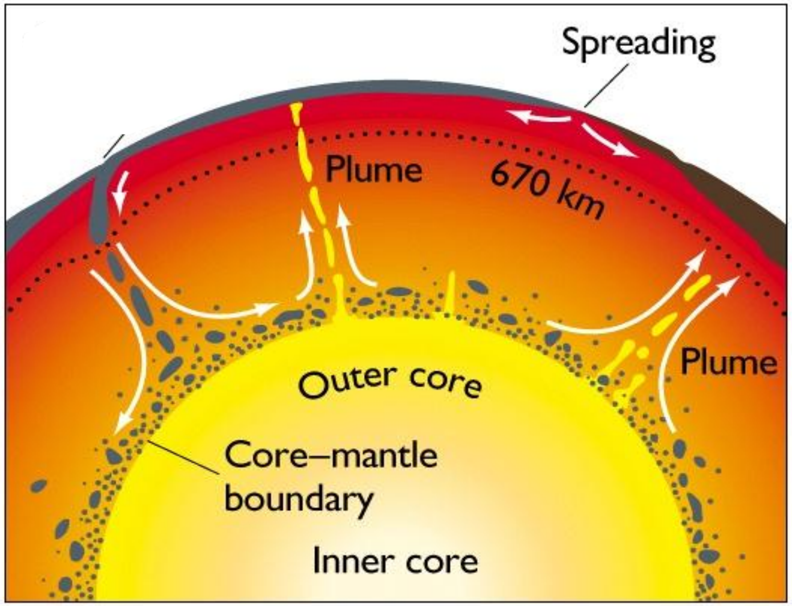
$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa \nabla^2 T + a$$

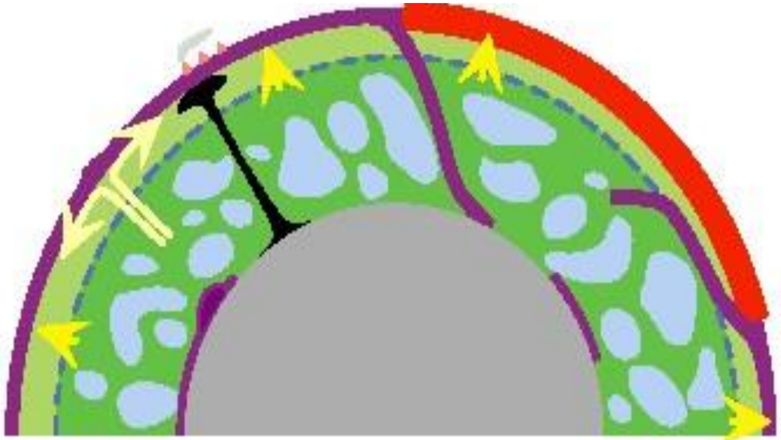
This time:

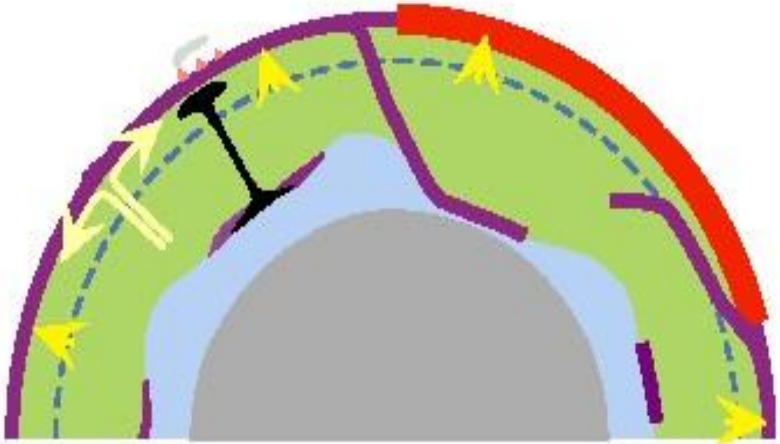
1) Check which students need Computer accounts on Geophysics computers
2) Sec. 6-19 Linear Stability Analysis for onset of convection—heated from below
3) Heating by Viscous Dissipation

4 Geoscientist's views of Earth's Interior



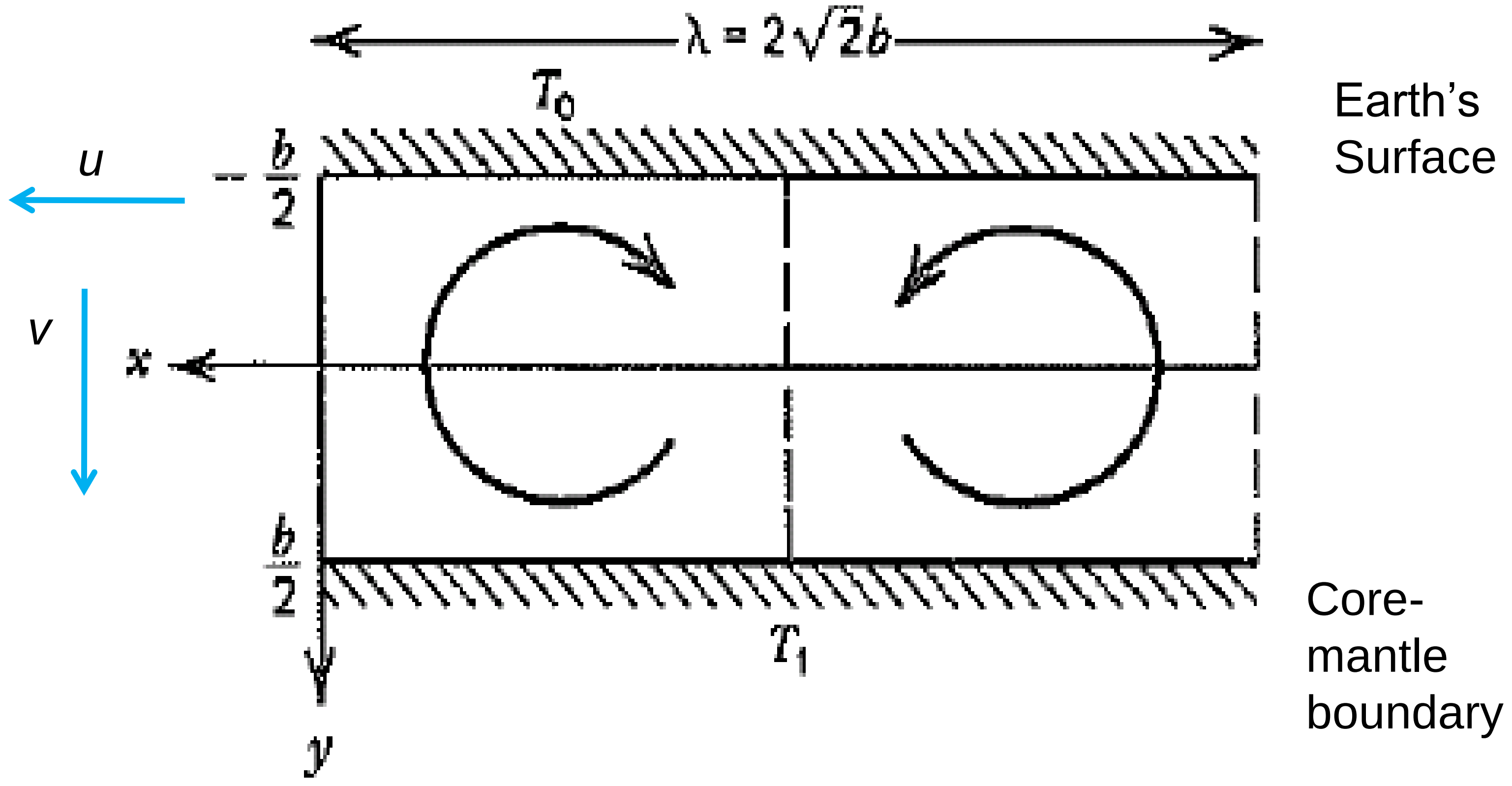






Linear Stability analysis for the onset of convection

6–38 Two-dimensional cellular convection in a fluid layer heated from below.



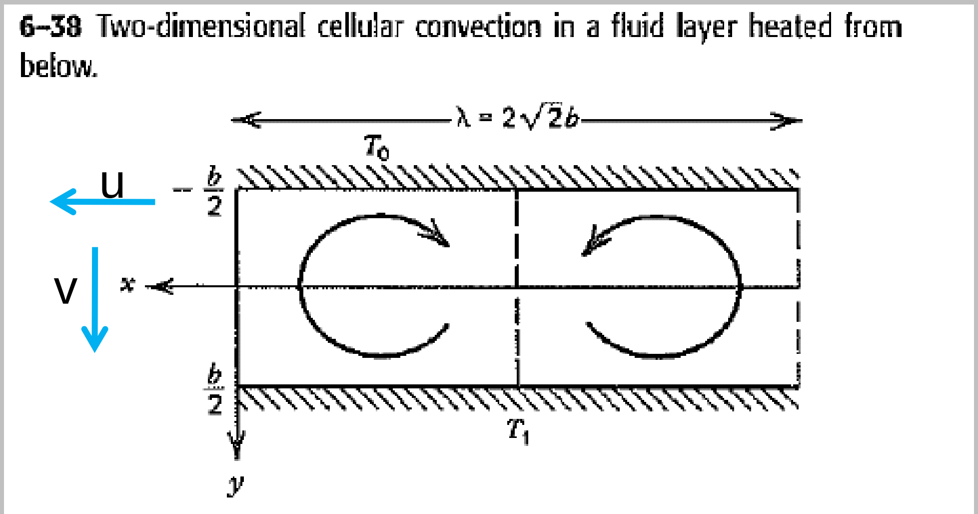
Linear Stability analysis for the onset of convection

- Buoyancy Force/Unit volume = $-g\rho_0\alpha_V(T-T_0)$
- Similar analysis to rise of diapir except density diff comes from heating➔
- Start with stable system (not moving)
- Heat up gradually until just when convection starts—allows approximations because movements are very small
- Define $T' = T - T_C$ =difference between actual Temp and Temp if only conduction occurred

Mantle flow animation

- Convection in the Earth's Mantle
- Higher temperature convection
- http://www.gps.caltech.edu/~gurnis/Movies/Animated_GIFs/slab401_movie.gif **(Superplume Formation Beneath An Ancient Slab)**
- Away from slab—plumes form rapidly and are small
- Under slab—plume takes longer to form and is large
- Slab buoyancey: Negative and blue; superplume buoyancy: red
- 3D convection

Heating from below



- $\boldsymbol{T_1 > T_0}$
- Assumptions:
- Start from $\boldsymbol{T = T_0}$
- Gradually heat until convection starts at $T = \boldsymbol{T_1}$ **(prime coordinates)**

$$u' = v' = 0$$

So just **before** convection:

$$\frac{\partial T_C}{\partial t} = 0$$

- (T_C is conduction solution)

$$\frac{\partial T_C}{\partial x} = 0$$

Heating from below: Conduction before convection

$$u' = v' = 0$$

$$\frac{\partial T_C}{\partial t} = 0$$

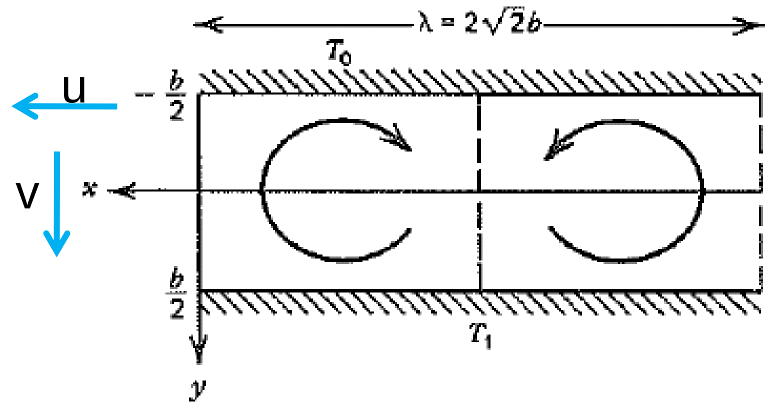
$$\frac{\partial T_C}{\partial x} = 0$$

$$\frac{\partial T_C}{\partial t} + \vec{u}' \cdot \vec{\nabla} T_C = \kappa \nabla^2 T_C + a$$

- (T_C is conduction solution)

$$\frac{\partial T_C}{\partial t} + u' \frac{\partial T_C}{\partial x} + v' \frac{\partial T_C}{\partial y} = \kappa \left(\frac{\partial^2 T_C}{\partial x^2} + \frac{\partial^2 T_C}{\partial y^2} \right) \quad (6-293)$$

6–38 Two-dimensional cellular convection in a fluid layer heated from below.



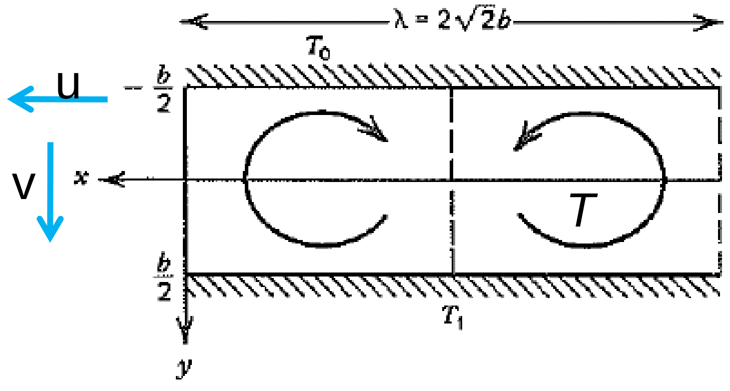
$$\therefore$$

$$\frac{\partial^2 T_C}{\partial y^2} = 0$$

Boundary Conditions

- For this case:
- Fluid flow alone (as seen in diapir analysis):
  - No-slip (Solid-Liquid) *u*=fixed=0 at *y*=+ *b/2*
  - Free surface: 0-stress (τ=0 ➔ $\delta u/\delta y$=0 1-d at $y=-b/2$)
- Heat: Isothermal: *T* continuous across boundary : $T=T_0$ at $y=-b/2$ $T=T_1$ at $y=+b/2$

6–38 Two-dimensional cellular convection in a fluid layer heated from below.



Just before convection

- Boundary conditions:
- $T = T_0$ at $y = -b/2$
- $T = T_1$ at $y = +b/2$
- Solution to: $$\frac{\partial^2 T_C}{\partial y^2} = 0$$

$$T_C = \frac{T_1 + T_0}{2} + \frac{T_1 - T_0}{b} y$$

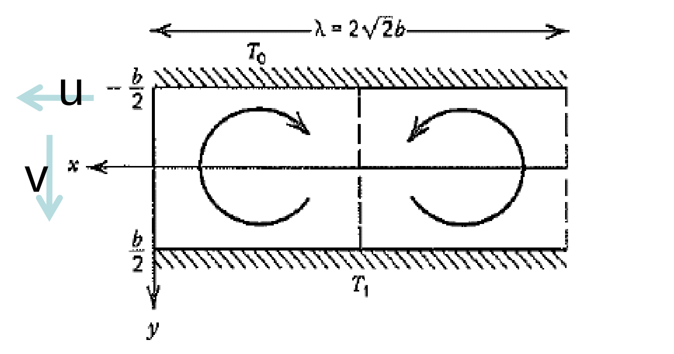
- (Linear temp. profile from top to bottom)

$$u' = v' = 0$$

$$\frac{\partial}{\partial t} = 0 \qquad \left(\frac{\partial T}{\partial t} = 0\right)$$

$$\frac{\partial}{\partial x} = 0 \qquad \left(\frac{\partial T}{\partial x} = 0\right)$$

6–38 Two-dimensional cellular convection in a fluid layer heated from below.



Just as convection starts

$$T' = T - T_C = T - \left(\frac{T_1 + T_0}{2} - \frac{T_1 - T_0}{b}\right) y$$

- *T'* is very small = departure of fluid temp. from conductivity profile $T' \approx u' \approx v' \approx 0$
- (solve for *T'*—easier)
- Small things: $\frac{\partial T'}{\partial t} \approx 0 \approx \frac{\partial T'}{\partial x} \approx 0$
- Even smaller things (products of small things): $u' \frac{\partial T'}{\partial x} \approx v' \frac{\partial T'}{\partial y}$

Equations reduce to:

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa \nabla^2 T + a$$

a=0

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$$

6-299

$$0 = -\frac{\partial P'}{\partial x} + \mu\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2}\right)$$

6-300 (from 6-64 & 6-67: fluid flow Sec. 6-8)

$$0 = -\frac{\partial P'}{\partial y} - \rho_0 \alpha_v g T' + \mu\left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2}\right)$$

6-301 (from 6-65)

$$\frac{\partial T'}{\partial t} + \frac{v'}{b}(T_1 - T_0) = \kappa\left(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2}\right)$$

6-302

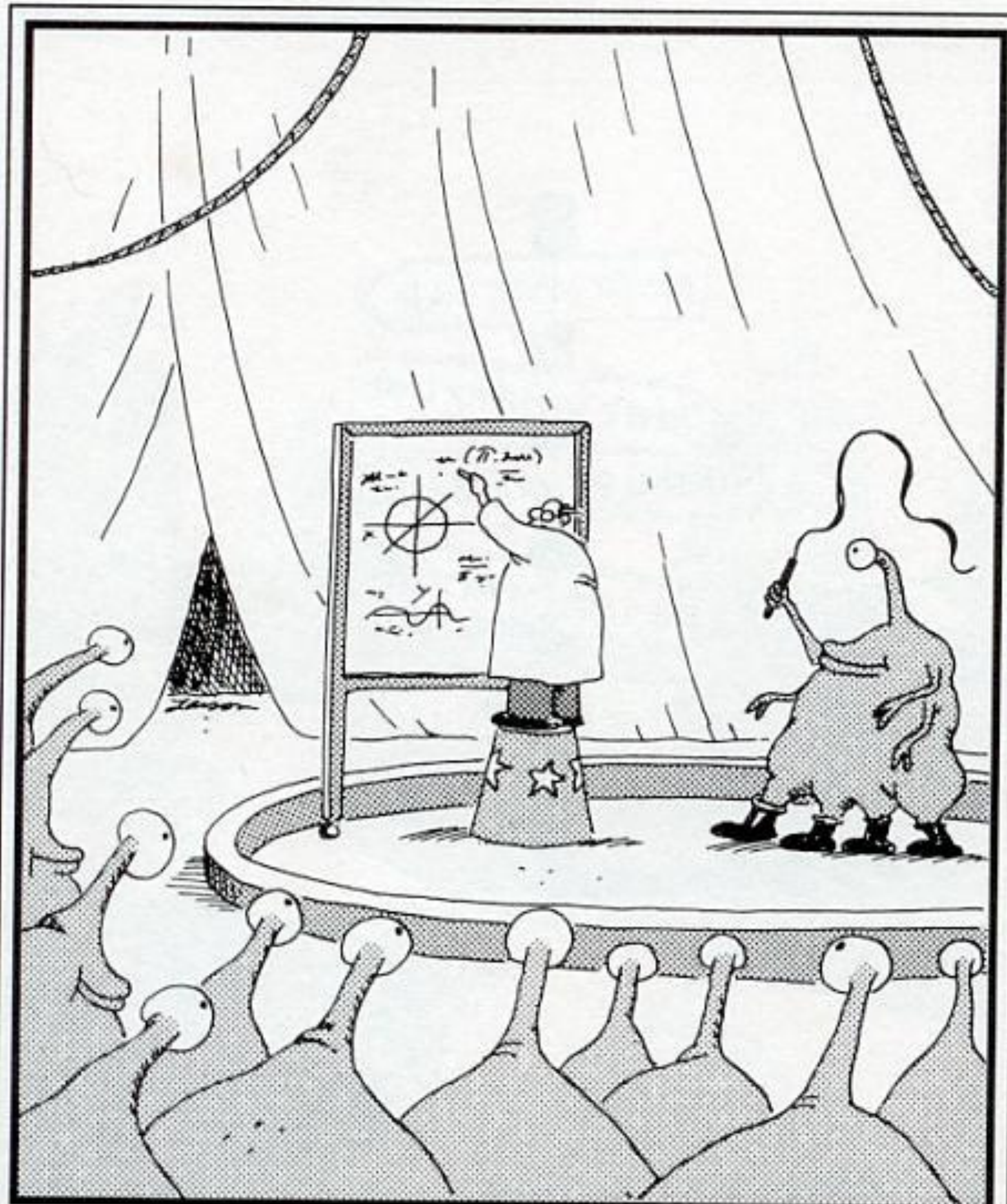
Introducting stream function, Equations to solve reduce to two coupled diff. eqns:

$$u = -\frac{\partial \psi}{\partial y}; v = \frac{\partial \psi}{\partial x}$$

$$\frac{\partial T'}{\partial t} + \frac{1}{b}(T_1 - T_0)\frac{\partial \psi'}{\partial x} = \kappa\left(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2}\right)(6-309)$$

$$0 = \mu\left(\frac{\partial^4 \psi'}{\partial x^4} + 2\frac{\partial^4 \psi'}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi'}{\partial y^4}\right) - \rho_0 g \alpha_v \frac{\partial T'}{\partial x}(6-310)$$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa \nabla^2 T + a$$



Abducted by an alien circus company, Professor Doyle is forced to write calculus equations in center ring.

Solution: Use Separation of Variables (y, x, t independent)

$$\psi' = \psi'_0 \cos\frac{(\pi y)}{b}\sin(\frac{2\pi x}{\lambda})e^{\alpha' t} \quad (6-311)$$

$$T' = T'_0 \cos\frac{(\pi y)}{b}\cos(\frac{2\pi x}{\lambda})e^{\alpha' t} \quad (6-312)$$

$$u' = -\frac{\partial \psi'}{\partial y}; v' = \frac{\partial \psi'}{\partial x}$$

α'=growth rate. If >0, get unstable growth➔ convection
If $\alpha' < 0$, decays with time

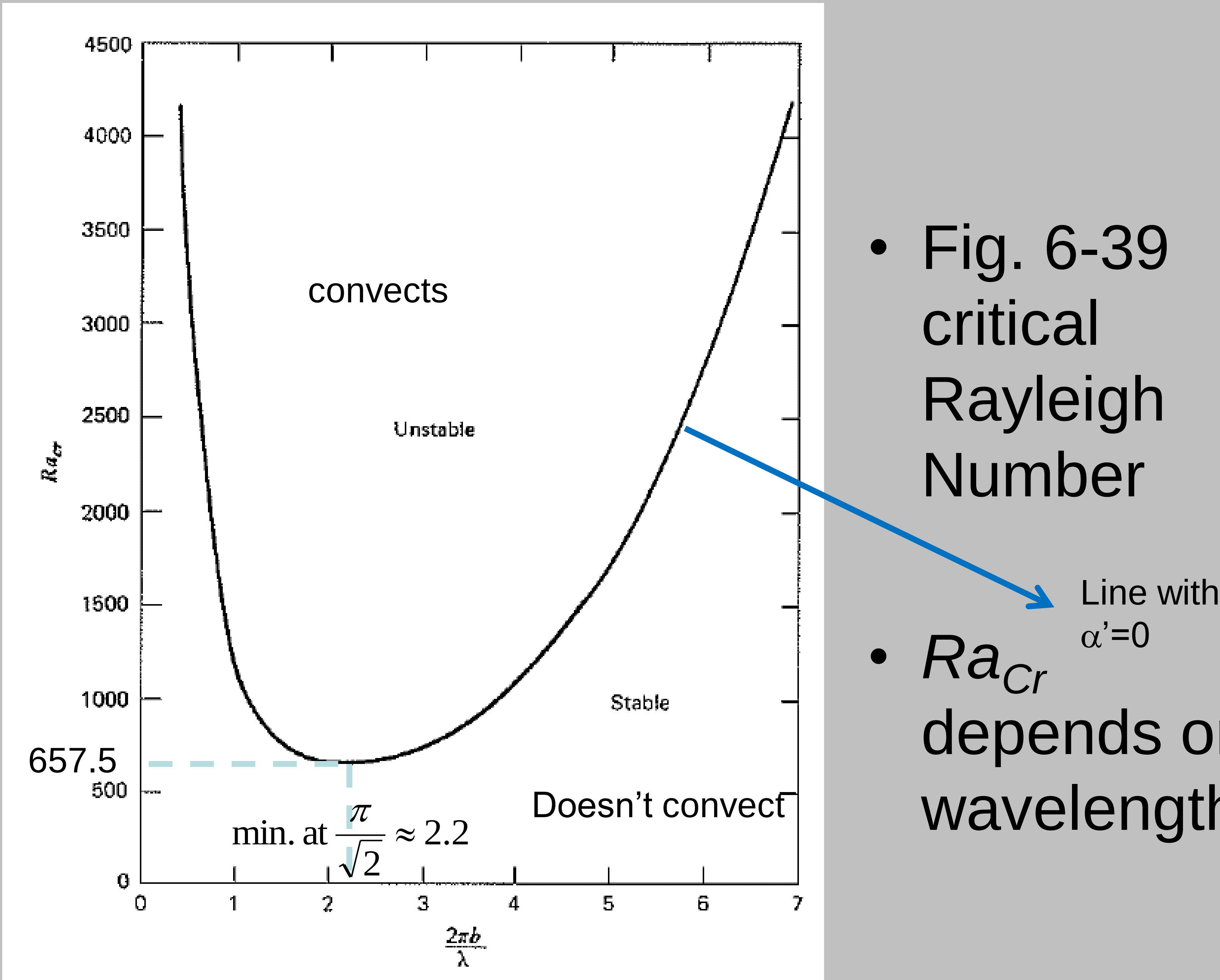
Substituting in values, get:
α'=*Ra*(function($2\pi b/\lambda$)), where $2\pi b/\lambda$=dimensionless wavenumber and
Ra= Rayleigh number, another dimensionless number

Rayleigh Number, Ra

- Dimensionless
- If *Ra* > (some large value), material convects

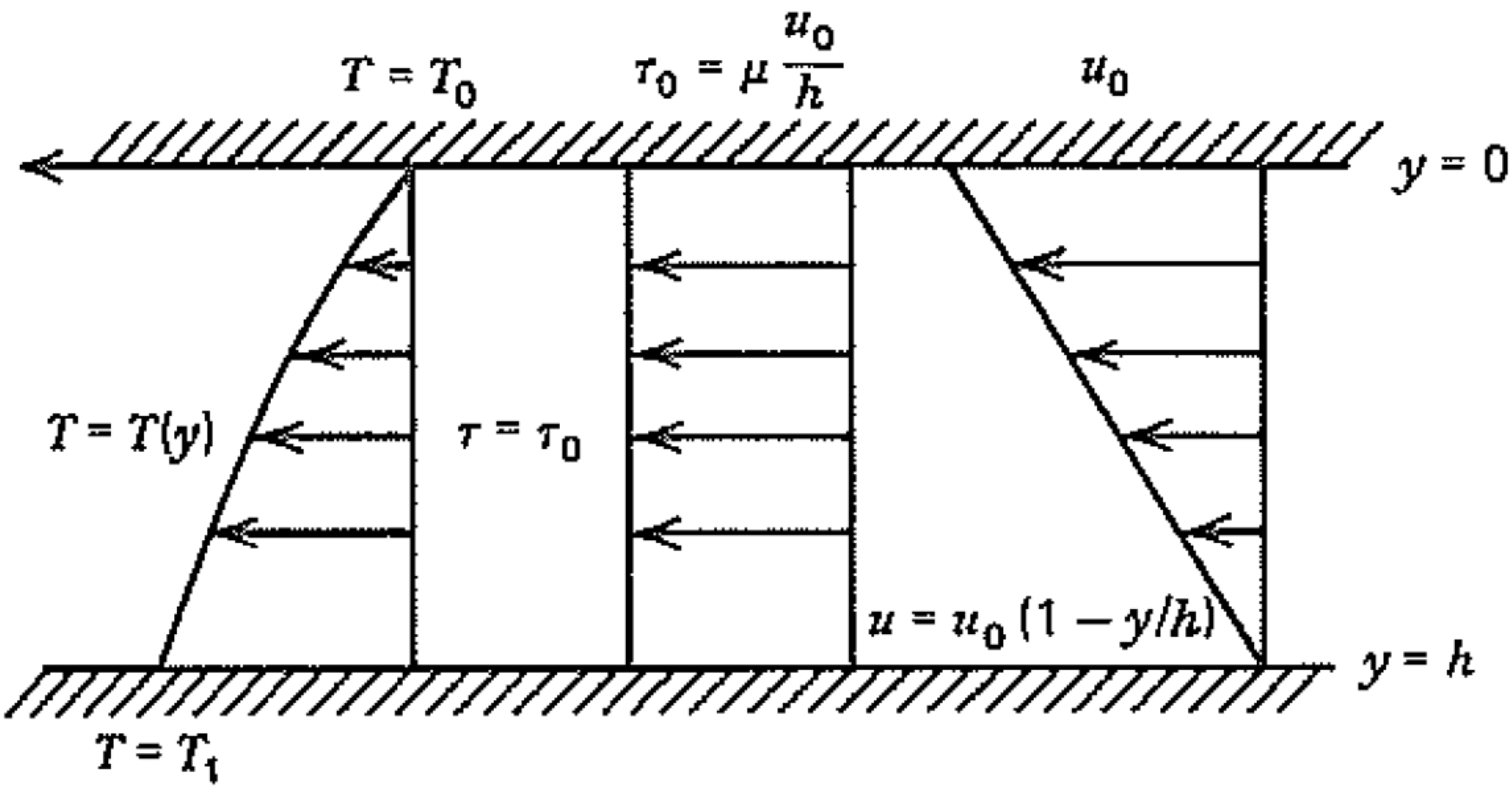
$$Ra = \frac{g\rho\alpha_V (T_1 - T_0) b^3}{\kappa\mu}$$

- (factors aiding convection)
- Factors inhibiting convection



- Fig. 6-39 critical Rayleigh Number
- Ra_{Cr} depends on wavelength

6-23: Heating by Viscous Dissipation



6–45 Frictional heating in Couette flow.

Rate of work

- Work =Force x distance
- Rate of work = Force x distance/time
- Stress = Force/area
- So Rate of work/unit area = Stress x distance/time = stress x velocity
- Rate of work/horiz. Area = shear stress x velocity
- (Book says—work on entire layer is given by stress and velocity at the top layer)

Another derivation

To get work done on the entire fluid layer per horizontal area—un-numbered equation on p. 283

$$work = \int \sigma_{ij} \, \tau_{ij} \, d\tau$$

$$= \int_h^0 \frac{\mu u_0}{h} \left(\frac{du}{dy}\right) dy = \int_h^0 \frac{\mu u_0}{h} du$$

$$= \frac{\mu u_0}{h} u \Big|_{u_0}^{0}$$

$$= \frac{-\mu u_0{}^2}{h}$$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa \nabla^2 T + a$$

Steady state: no change with time

Also velocity ⊥grad(T)

- So the shear heating is the volumetric heat production (*ρH*) and we get

$$k \frac{d^2T}{dy^2} = \rho H = -\frac{\mu u_0{}^2}{h^2}$$

- Get the temperature distribution in dimensionless form

$$\theta = \frac{T - T_0}{T_1 - T_0}$$

- Depends on another dimensionless parameter-the Eckert number

Eckert number

$$E \equiv \frac{{u_0}^2}{c_p(T_1 - T_0)}$$

Where c_p is specific heat at constant pressure.

Final solution depends on the product of two dimensionless numbers, Prandtl number *Pr* and Eckert number *E*

PrE

Problem Hint

- Don’t forget boundary condition
- q=0 across boundary ➔ $\delta T/\delta y=0$
- $Ra<Ra_{Cr}$ ➔ no convection
- $Ra>Ra_{Cr}$ ➔ yes convection

Problem hints

- Some given in the assignment handout. Particularly, check misprints.