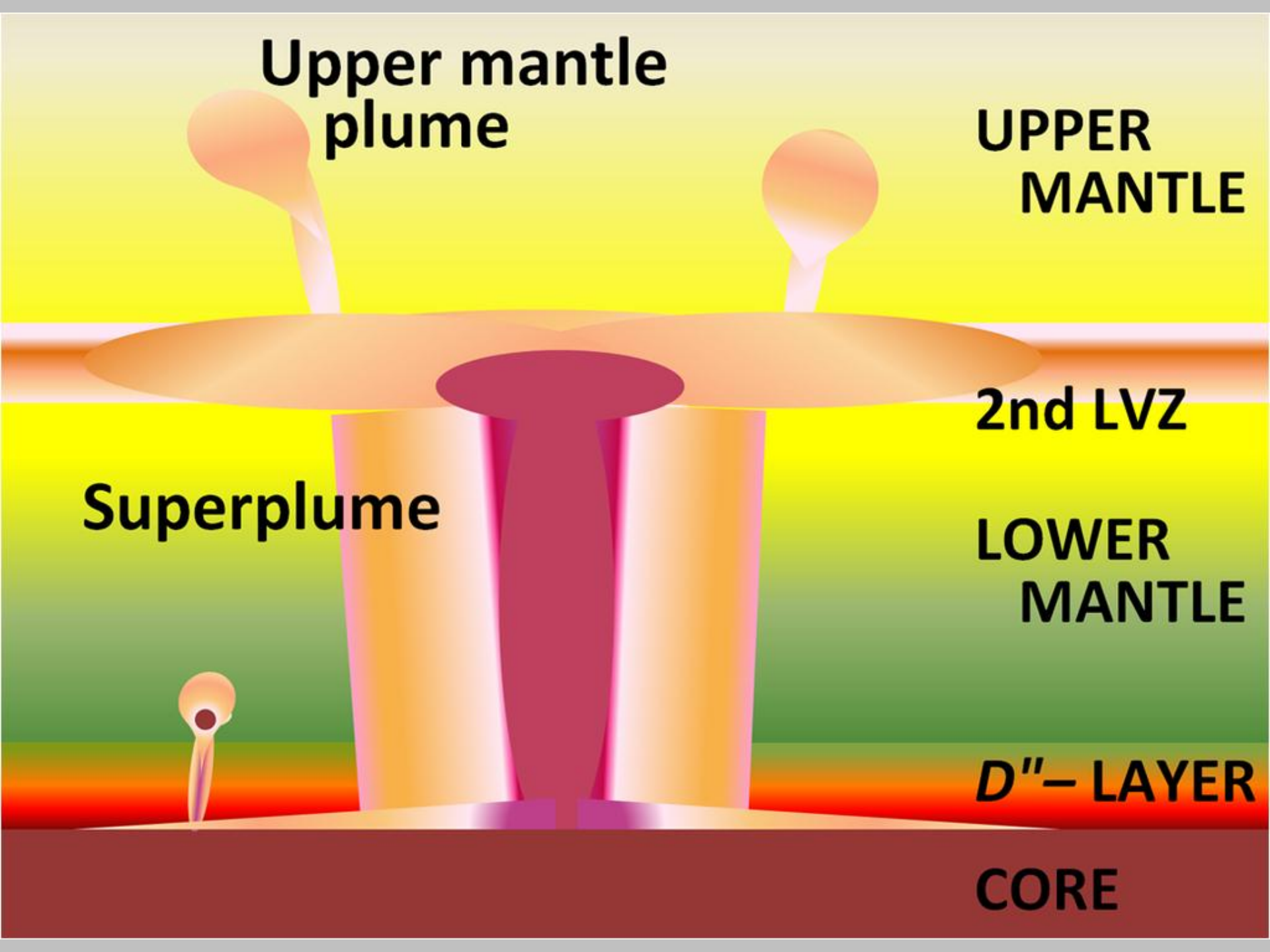


Notes for Assignment 3

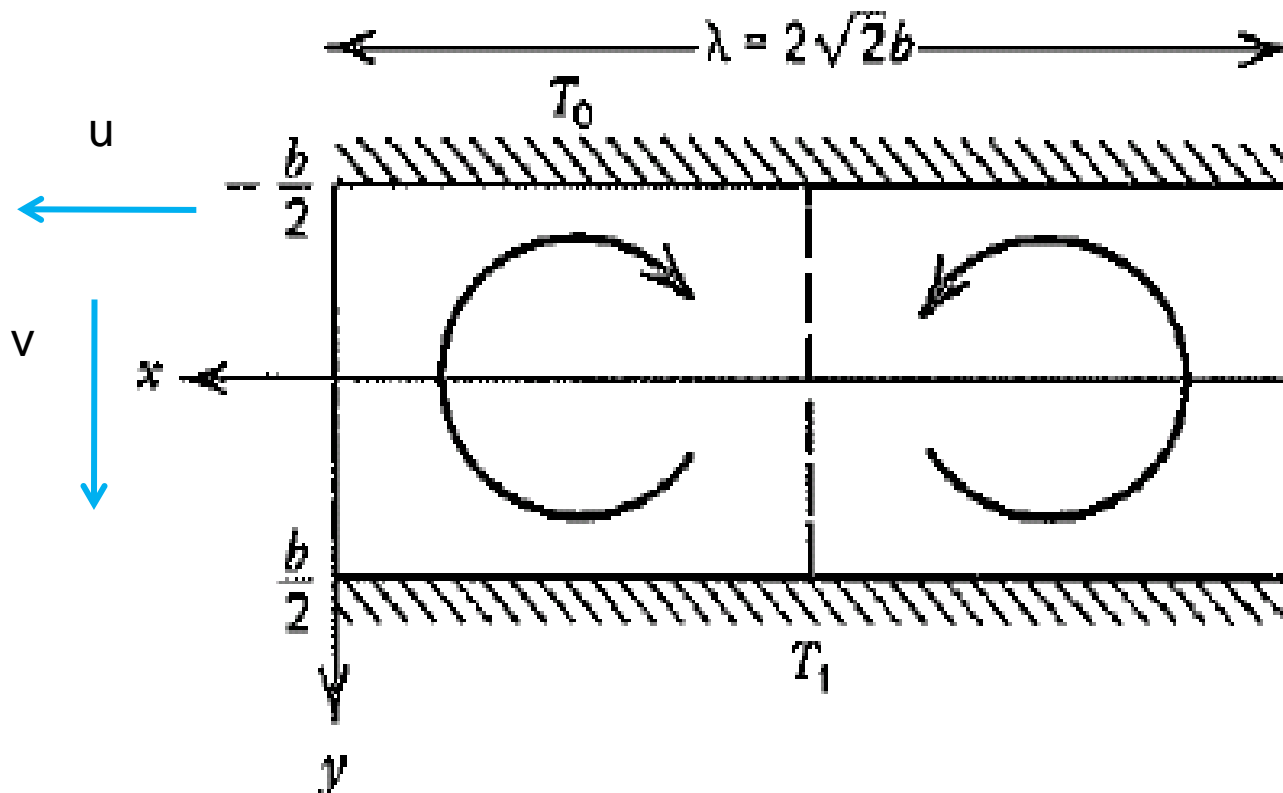
Maths 323 Fluids Module 2014

- 1) Concepts from Turcotte & Schubert Ch. 4 needed in Ch. 6
- 2) Thermal expansion and Plume Heads and Tails (Section 6-15)
- 3) Thermal Convection
- 4) Convection Movies
- 5) Sec. 6-19 Linear Stability Analysis for onset of convection—heated from below
- 6) Heating by Viscous Dissipation



Linear Stability analysis for the onset of convection

6-38 Two-dimensional cellular convection in a fluid layer heated from below.



Linear Stability analysis for the onset of convection

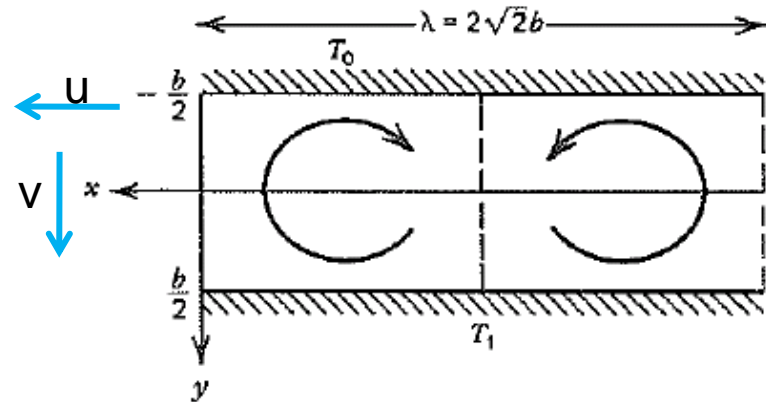
- Buoyancy Force/Unit volume = $-g\rho_0\alpha_V(T-T_0)$
- Similar analysis to rise of diapir except density diff comes from heating →
- Start with stable system (not moving)
- Heat up gradually until just when convection starts—allows approximations because movements are very small
- Define $T'=T-T_C$ =difference between actual Temp and Temp if only conduction occurred

Mantle flow animation

- [Convection in the Earth's Mantle](#)
- [Higher temperature convection](#)
- http://www.gps.caltech.edu/~gurnis/Movies/Animated_GIFs/slab401_movie.gif (**Superplume Formation Beneath An Ancient Slab**)
- Away from slab—plumes form rapidly and are small
- Under slab—plume takes longer to form and is large
- Slab buoyancy: Negative and blue; superplume buoyancy: red
- [3D convection](#)

Heating from below

6-38 Two-dimensional cellular convection in a fluid layer heated from below.



- $T_1 > T_0$
- Assumptions:
- Start from $T = T_0$
- Gradually heat until convection starts at $T = T_1$

$$u = v = 0$$

So just **before** convection:

$$\frac{\partial T}{\partial t} = 0; \frac{\partial T_c}{\partial t} = 0$$

- (T_c is conduction solution)

$$\frac{\partial T}{\partial x} = 0; \frac{\partial T_c}{\partial x} = 0$$

Heating from below conduction before convection

- (T_C is conduction solution)

$$u = v = 0$$

$$\frac{\partial T}{\partial t} = 0$$

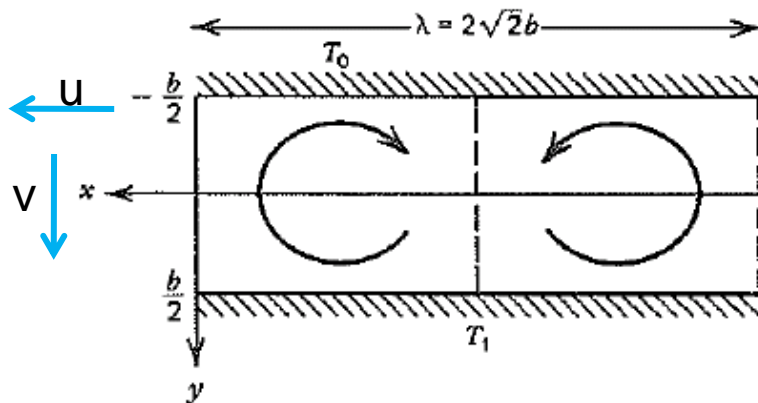
$$\frac{\partial T}{\partial x} = 0$$

~~$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \quad (6-293)$$~~

∴

$$\frac{\partial^2 T_C}{\partial y^2} = 0$$

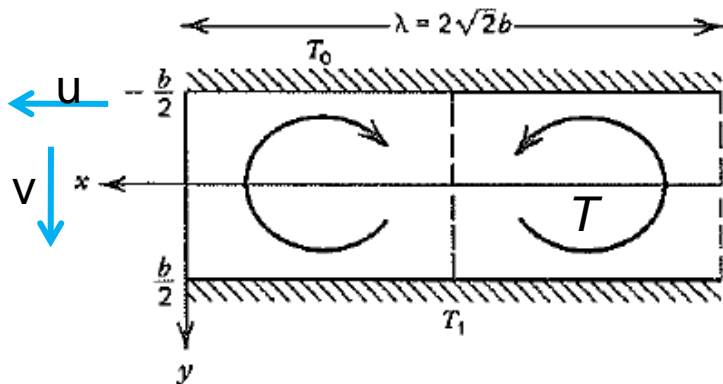
6-38 Two-dimensional cellular convection in a fluid layer heated from below.



Boundary Conditions

- For this case:
- Fluid flow alone :
 - No-slip (Solid-Liquid) $u = \text{fixed} = 0$ at $y = +b/2$
 - Free surface: 0-stress ($\tau = 0 \rightarrow \delta u / \delta y = 0$ 1-d at $y = -b/2$)
- Heat: Isothermal: T continuous across boundary : $T = T_0$ at $y = -b/2$ $T = T_1$ at $y = +b/2$

6-38 Two-dimensional cellular convection in a fluid layer heated from below.



Just before convection

- Boundary conditions:

$$u' = v' = 0$$

- $T = T_0$ at $y = -b/2$

$$\frac{\partial}{\partial t} = 0 \quad \left(\frac{\partial T}{\partial t} = 0 \right)$$

- $T = T_1$ at $y = +b/2$

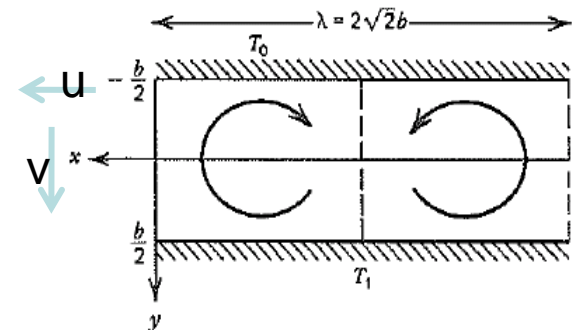
- Solution to: $\frac{\partial^2 T_c}{\partial y^2} = 0$

$$\frac{\partial}{\partial x} = 0 \quad \left(\frac{\partial T}{\partial x} = 0 \right)$$

$$T_c = \frac{T_1 + T_0}{2} + \frac{T_1 - T_0}{b} y$$

- (Linear temp. profile from top to bottom)

6-38 Two-dimensional cellular convection in a fluid layer heated from below.



Just as convection starts

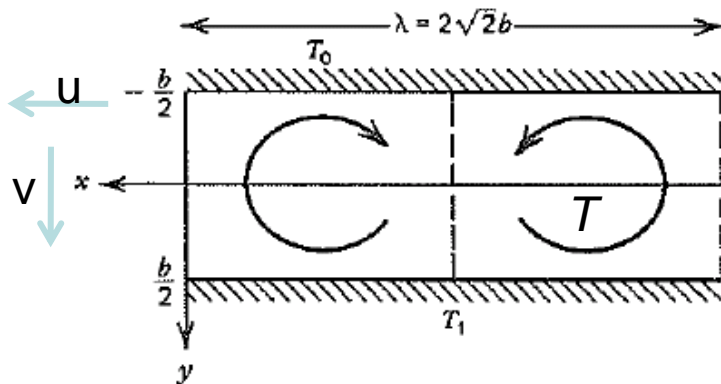
$$T' = T - T_c = T - \left(\frac{T_1 + T_0}{2} - \frac{T_1 - T_0}{b} \right) y$$

- T' is very small = departure of fluid temp. from conductivity profile $T' \approx u' \approx v' \approx 0$
- (solve for T' —easier) $\frac{\partial T'}{\partial t} \approx 0 \approx \frac{\partial T'}{\partial x} \approx 0$
- Small things:
- Even smaller things (products of small things): $u' \frac{\partial T'}{\partial x} \approx v' \frac{\partial T'}{\partial y}$

Boundary Conditions

- For this case:
- Fluid flow alone (as seen in diapir analysis):
 - No-slip (Solid-Liquid) $u = \text{fixed} = 0$ at $y = +b/2$
 - Free surface: 0-stress ($\tau = 0 \rightarrow \delta u / \delta y = 0$ 1-d at $y = -b/2$)
- Heat: Isothermal: T continuous across boundary : $T = T_0$ at $y = -b/2$ $T = T_1$ at $y = +b/2$

6-38 Two-dimensional cellular convection in a fluid layer heated from below.



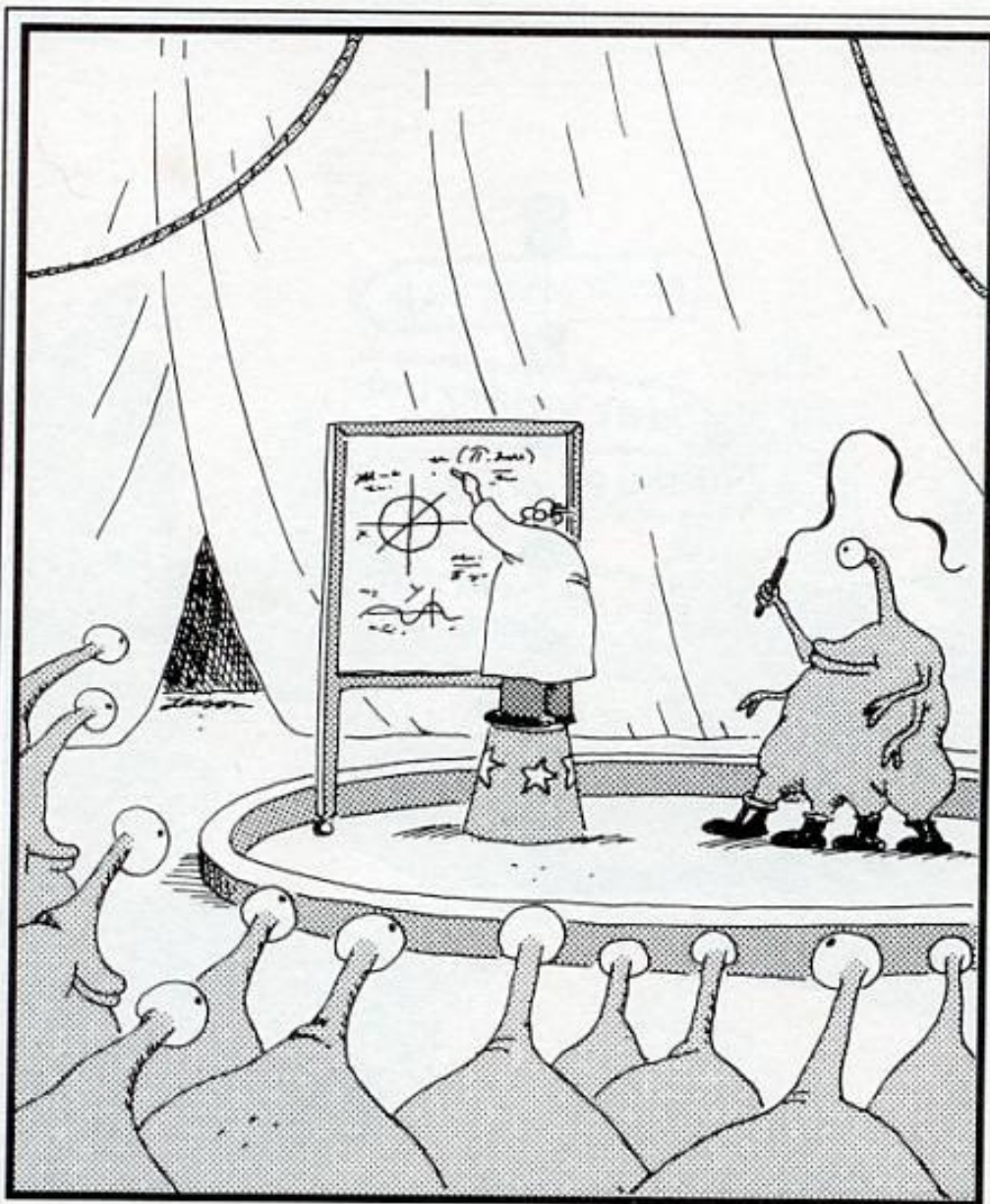
Introducing stream function,
 Equations to solve reduce to
 two coupled diff. eqns:

$$u = -\frac{\partial \psi}{\partial y}; v = \frac{\partial \psi}{\partial x}$$

$$\frac{\partial T'}{\partial t} + \frac{1}{b}(T_1 - T_0) \frac{\partial \psi'}{\partial x} = \kappa \left(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2} \right) \quad (6-309)$$

$$0 = \mu \left(\frac{\partial^4 \psi'}{\partial x^4} + 2 \frac{\partial^4 \psi'}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi'}{\partial y^4} \right) - \rho_0 g \alpha_v \frac{\partial T'}{\partial x} \quad (6-310)$$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa \nabla^2 T + a$$



Abducted by an alien circus company, Professor Doyle is forced to write calculus equations in center ring.

Solution: Use Separation of Variables (y, x, t independent)

$$\psi' = \psi'_0 \cos\left(\frac{\pi y}{b}\right) \sin\left(\frac{2\pi x}{\lambda}\right) e^{\alpha' t} \quad (6-311)$$

$$T' = T'_0 \cos\left(\frac{\pi y}{b}\right) \cos\left(\frac{2\pi x}{\lambda}\right) e^{\alpha' t} \quad (6-312)$$

$$u' = -\frac{\partial \psi'}{\partial y}; v' = \frac{\partial \psi'}{\partial x}$$

α' =growth rate. If >0 , get unstable growth \rightarrow convection
If $\alpha' < 0$, decays with time

Substituting in values, get:

$\alpha' = Ra(\text{function}(2\pi b/\lambda))$, where $2\pi b/\lambda$ =dimensionless wavenumber and
 Ra = Rayleigh number, another dimensionless number

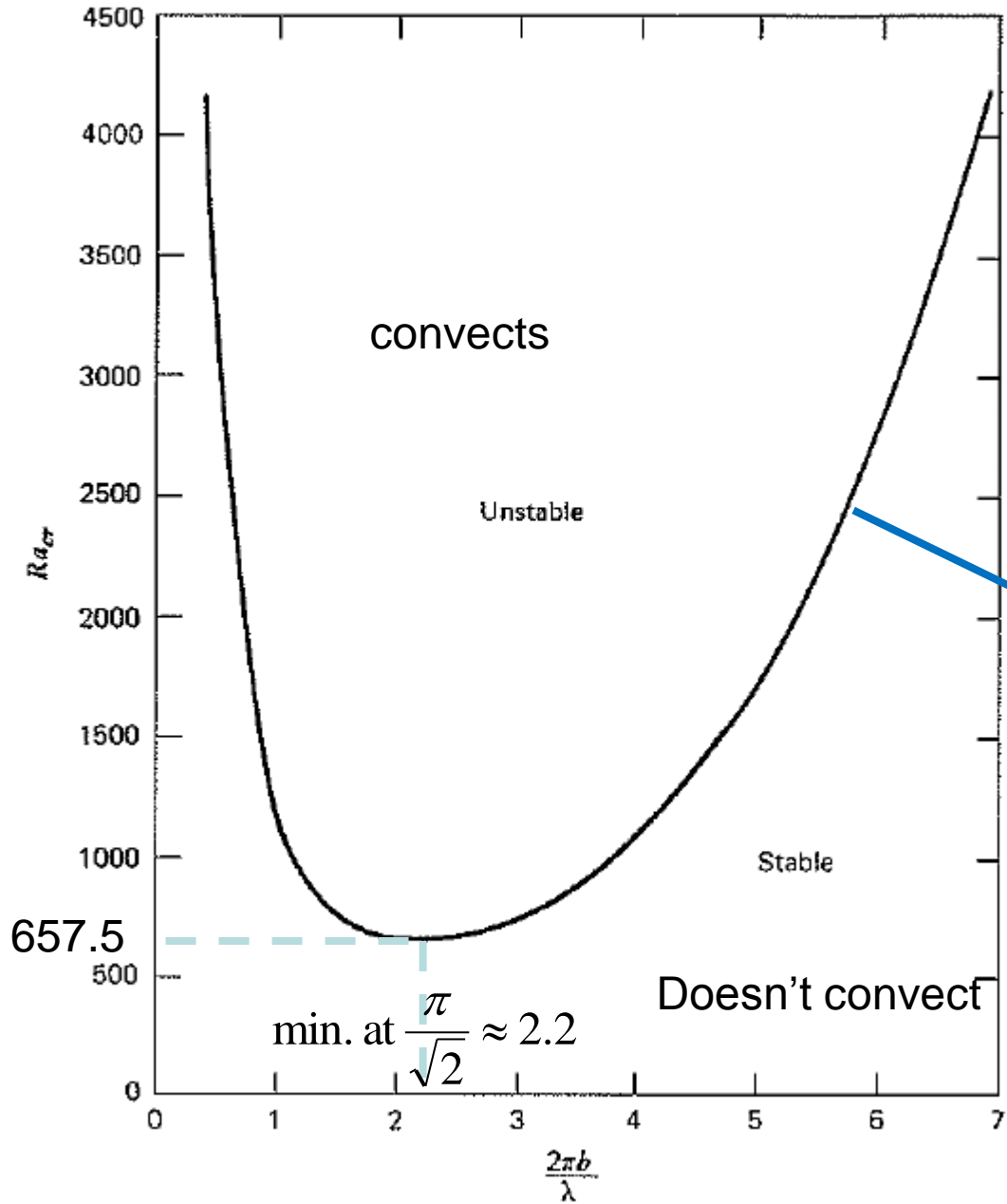
Rayleigh Number, Ra

- Dimensionless
- If $Ra >$ (some large value), material convects

$$Ra = \frac{g\rho\alpha_V(T_1 - T_0)b^3}{\kappa\mu}$$

■(factors aiding convection)

■Factors inhibiting convection

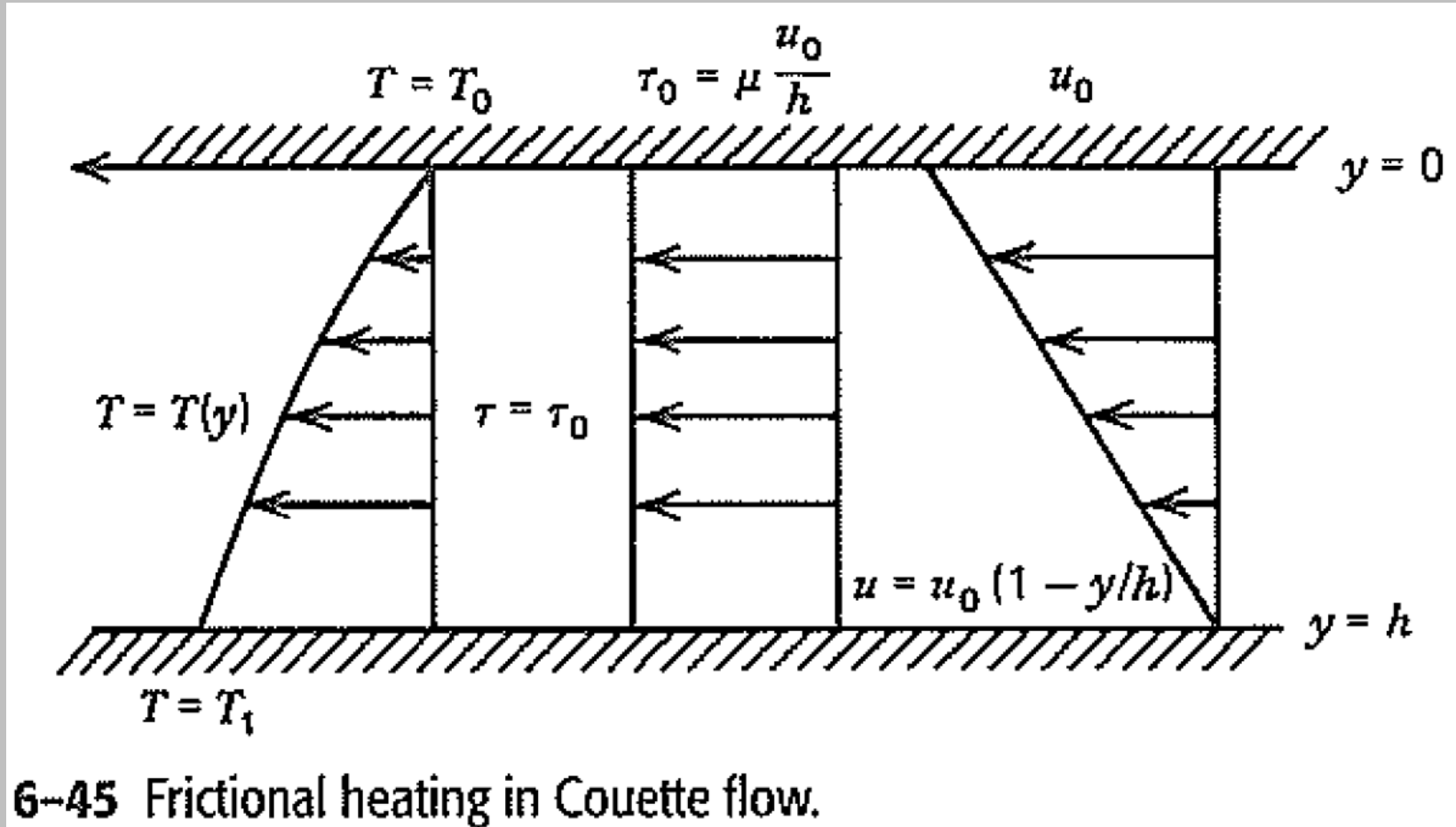


- Fig. 6-39 critical Rayleigh Number

- Ra_{Cr} depends on wavelength

Line with $\alpha'=0$

6-23: Heating by Viscous Dissipation



Rate of work

- $\text{Work} = \text{Force} \times \text{distance}$
- $\text{Rate of work} = \text{Force} \times \text{distance}/\text{time}$
- $\text{Stress} = \text{Force}/\text{area}$
- So $\text{Rate of work}/\text{unit area} = \text{Stress} \times \text{distance}/\text{time} = \text{stress} \times \text{velocity}$
- $\text{Rate of work}/\text{horiz. Area} = \text{shear stress} \times \text{velocity}$
- (Book says—work on entire layer is given by stress and velocity at the top layer)

Another derivation

To get work done on the entire fluid layer per horizontal area—un-numbered equation on p. 283

$$\begin{aligned} \text{work} &= \int \sigma_{ij} \tau_{ij} d\tau \\ &= \int_h^0 \frac{\mu u_0}{h} \left(\frac{du}{dy} \right) dy = \int_h^0 \frac{\mu u_0}{h} du \\ &= \frac{\mu u_0}{h} u \Big|_{u_0}^0 \\ &= \frac{-\mu u_0^2}{h} \end{aligned}$$

- So the shear heating is the volumetric heat production (ρH) and we get

$$k \frac{d^2 T}{dy^2} = \rho H = -\frac{\mu u_0^2}{h^2}$$

- Get the temperature distribution in dimensionless form

$$\theta = \frac{T - T_0}{T_1 - T_0}$$

- Depends on another dimensionless parameter-the Eckert number

Eckert number

$$E \equiv \frac{u_0^2}{c_p (T_1 - T_0)}$$

Where c_p is specific heat at constant pressure.

Final solution depends on the product of two dimensionless numbers, Prandtl number and Eckert number

PrE

Problem Hint

- Don't forget boundary condition
- $q=0$ across boundary $\rightarrow \delta T / \delta y = 0$
- $Ra < Ra_{Cr} \rightarrow$ no convection
- $Ra > Ra_{Cr} \rightarrow$ yes convection

Problem hints

- Some given in the assignment handout. Particularly, check misprints.

Boundary conditions become:

- On $y=\pm b/2$ isothermal, no flow across them so
- $T'=v'=0$ for all $x \rightarrow \delta v'/\delta x=0$
- Use free surface conditions to get an analytic soln:
- At $y=\pm b/2$, $\delta u'/\delta y=0$
- ' denotes value at
- start of convection

6-38 Two-dimensional cellular convection in a fluid layer heated from below.

