Notes for Assignment 1
Gphs/Maths 323 2014
Fluid Flow in Earth Systems
Last time:

1) Definitions of fluid
2) Equivalence of strain rate and velocity gradient
3) Dimensional inconsistencies
4) Dimensionless numbers: Prandtl number
5) Boundary conditions:
   - Free surface (τ=0)
   - No-slip surface (velocity constant at boundary)

This time:

- Review quiz
- See me if you are enrolled in this as part of a 400-level course
- Detailed examples
- Definition of volumetric flow rate
- Asthenospheric counterflow
- Pipe flow
- Reynold’s number

Quiz review

- What is a fluid?
- What is a dimensionless number and why are they important in fluid mechanics?
- What are boundary conditions?
-

Boundary Conditions

- The set of conditions specified for behavior of the solution to a set of differential equations at the boundary of its domain. (American Heritage® Dictionary of the English Language, Fourth Edition copyright ©2000 by Houghton Mifflin Company)
- Physical laws usually govern what happens at the boundary between two media.
- Differ from initial conditions in that boundary conditions are usually set from physical principles and initial conditions are assumed or measured, and only used for time.

Example-last slide shown 1-D fluid flow

$$\frac{d\tau}{dy} = \frac{dp}{dx} (6-8)$$

Starting Equations (pressure gradient causes gradient in shear stress); Stress is viscosity times strain rate or velocity gradient

$$\tau = \mu \frac{du}{dy} (6-1)$$

Differentiate wrt y and substitute ➔

$$\frac{d\tau}{dy} = \mu \frac{d^2u}{dy^2} = \frac{dp}{dx}$$

Integrate once➔

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx} y + C_1$$

If there is a free surface: Use this eqn to evaluate C_1

$$(\frac{du}{dy} = \frac{\tau}{\mu} = \frac{1}{\mu}\frac{dp}{dx} y + C_1 \; (\tau = 0 \text{ at some y position})$$

$$u = \frac{1}{2\mu}\frac{dp}{dx} y^2 + C_1 y + C_2$$

Integrate twice: Use no-slip condition here ($u=u_0$ at some y position)

Example: free surface at y=0 and no-slip at $y=y_0$

$$\frac{d\tau}{dy} = \mu \frac{d^2u}{dy^2} = \frac{dp}{dx}$$

Integrate once➔

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx}y + C_1$$

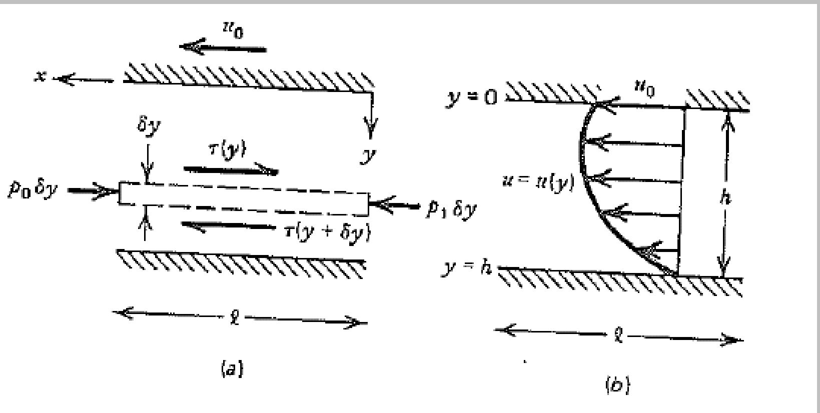
If there is a free surface: Use this eqn to evaluate C_1—example, if have free surface at y=0, then du/dy=τ/μ=0 at y=0, so $C_1=0$

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + C_1 y + C_2$$

Integrate twice: Use no-slip condition here
Example: $C_1=0$ and $u=u_0$ at $y=y_0$ then

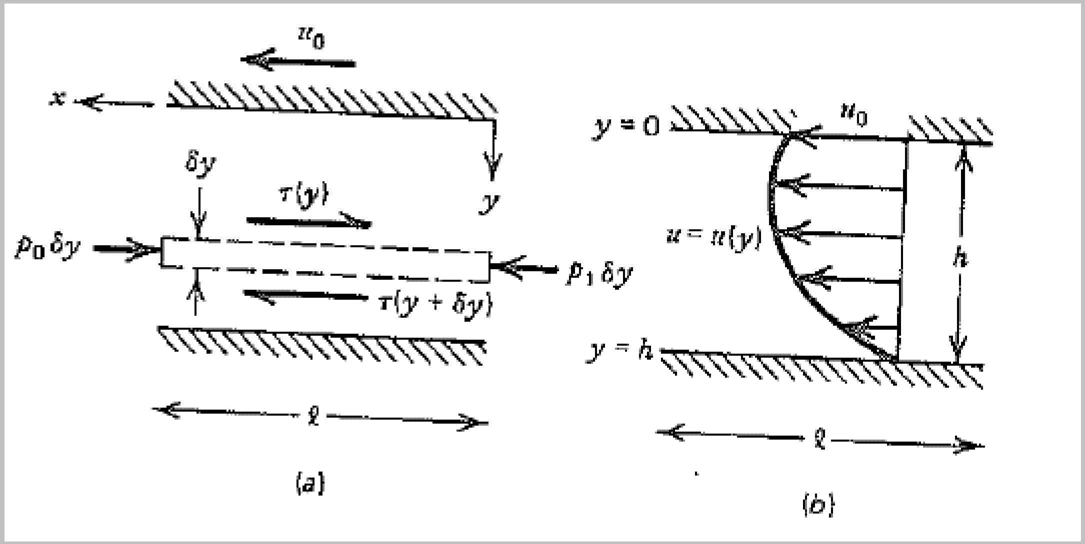
$$u_0 = \frac{1}{2\mu}\frac{dp}{dx}{y_0}^2 + C_2$$

$$C_2 = u_0 - \frac{1}{2\mu}\frac{dp}{dx}{y_0}^2$$



(This assumes that dp/dx is a known constant)

General solution for 1-D flow



- Equation 6-12: $$u = \frac{1}{2\mu}\frac{dp}{dx}(y^2 - hy) - \frac{u_0 y}{h} + u_0$$
- If dp/dx=0; Couette Flow
- If u=0, dp/dx≠0 no special name-just stationary boundary condition.

Example

- Also, get intermediate Solution: $\Delta P=\rho gH$
- Where H=Hydraulic head and ΔP=pressure difference:
- Difference in pressure depends only on height difference
- True for tubes—e.g., siphons, and also for reservoirs and water tanks.

Problem hints:

- What is the most important first step in solving an applied mathematics problem?

- Draw pictures!
- (first step is actually understanding the problem—drawing a picture helps enormously)
- Consider boundary conditions!

Problem hints

- Remember 1st year calculus—how do you get the average of a function?

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx$$

- How do you get the maximum and minimum of a function?

Set derivative equal to zero and check either side or look at second derivative (curvature) to see if positive (minimum) or negative (maximum) or—”cheat” by solving numerically and looking at graph.

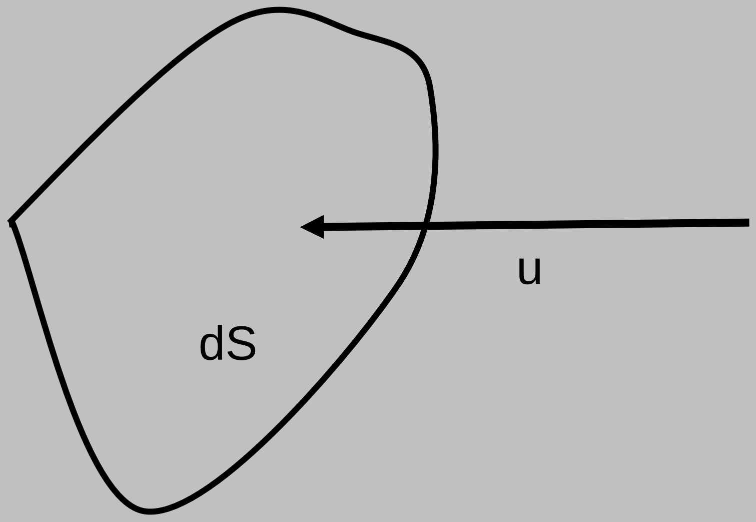
Hydraulic Head

- Pressure drops often defined by hydraulic head:
- $\Delta P = \rho g H$
- Where H=Hydraulic head and ΔP=pressure difference:
- Height H is the height of fluid required to provide the applied pressure difference purely hydrostatically.
- In the absence of outside forces, difference in pressure depends only on height difference (so if height is x, then $dp/dx = \rho g$)
- True for tubes—e.g., siphons, and also for reservoirs and water tanks and pressure inside the Earth

Volumetric flow rate

- Q = volumetric flow rate=total volume of fluid passing a cross-section per unit time.
- Examples: River, pipe

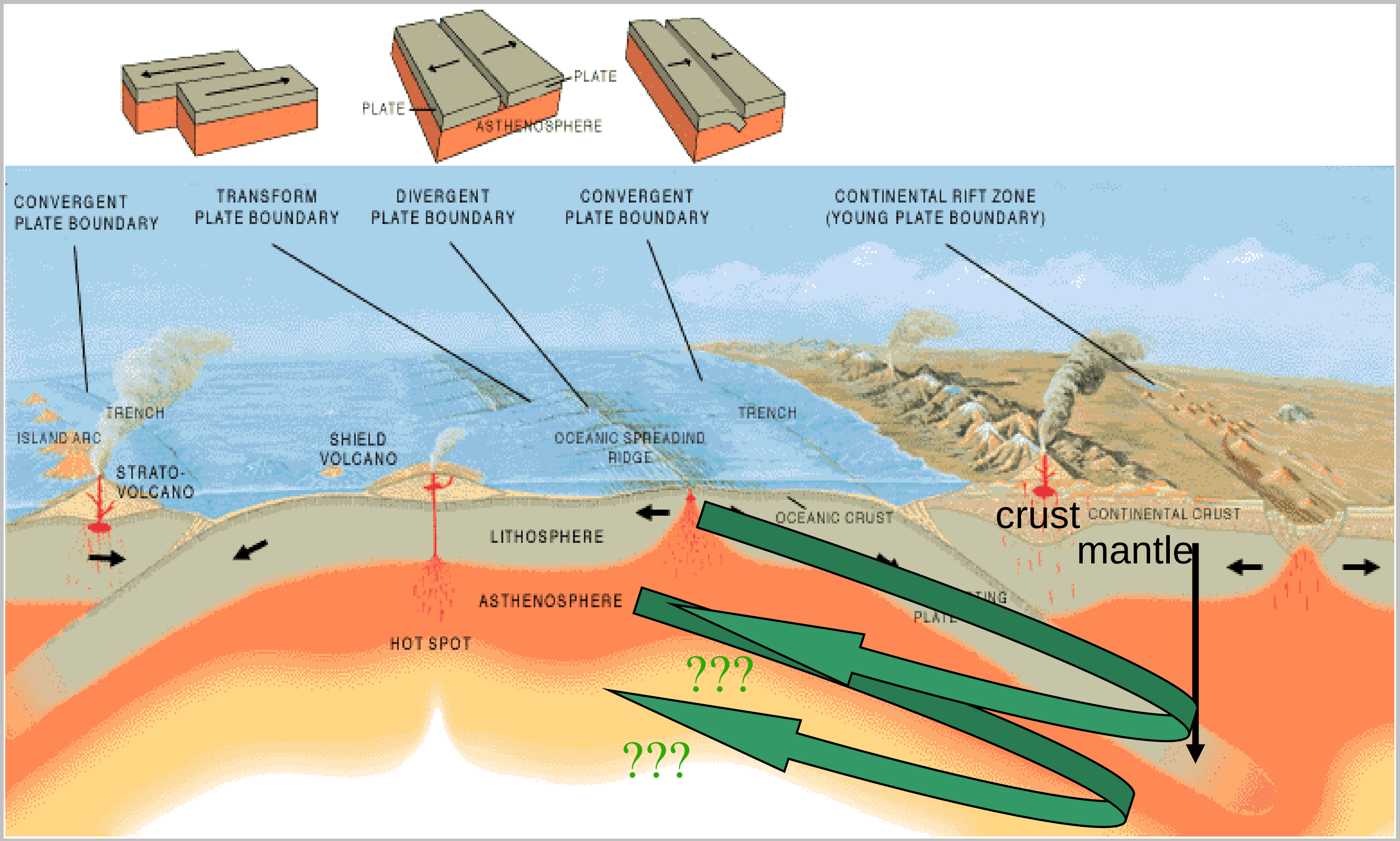
$$Q = \int_{surface} udS$$



- (u is component perp.

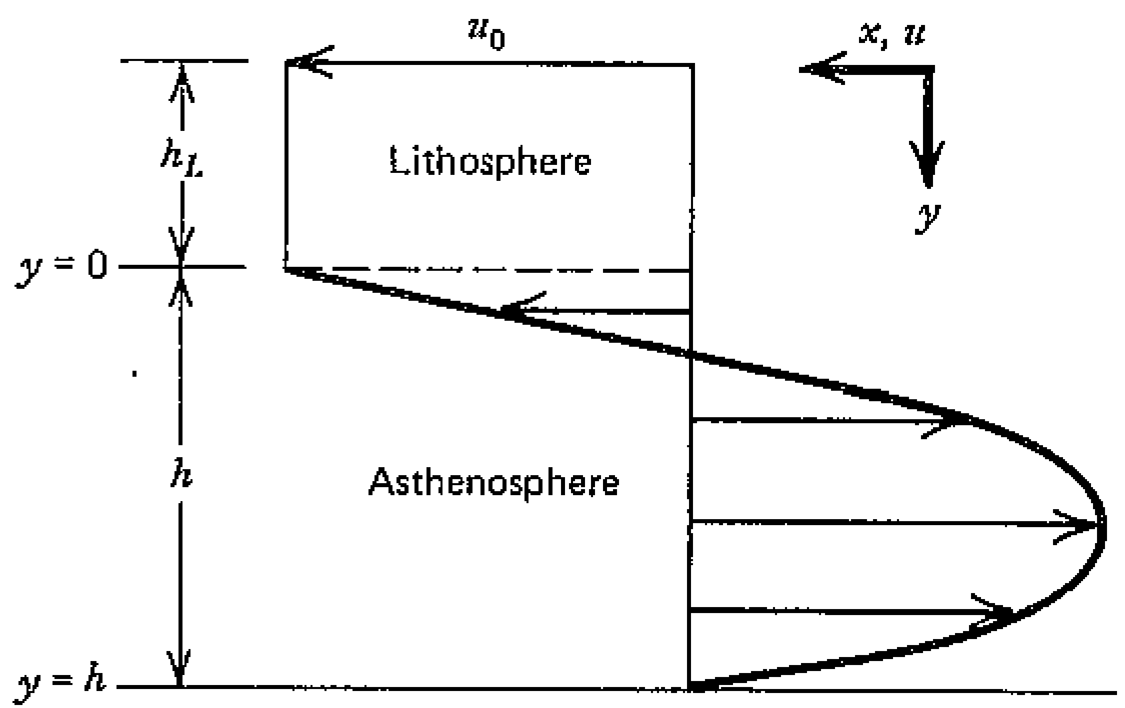
to surface)

Asthenospheric Counterflow



Asthenospheric counterflow

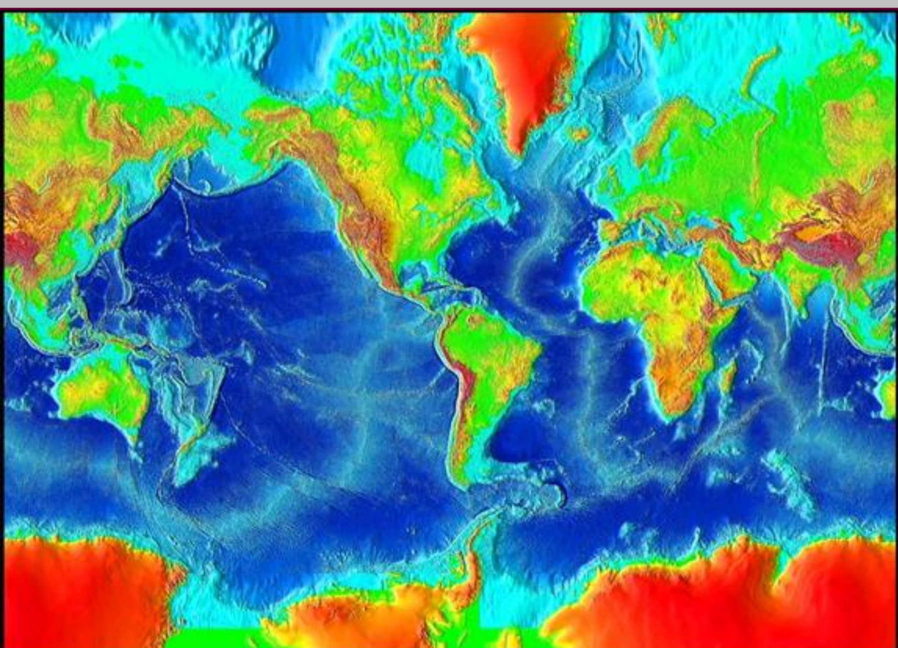
- People originally thought it might exist

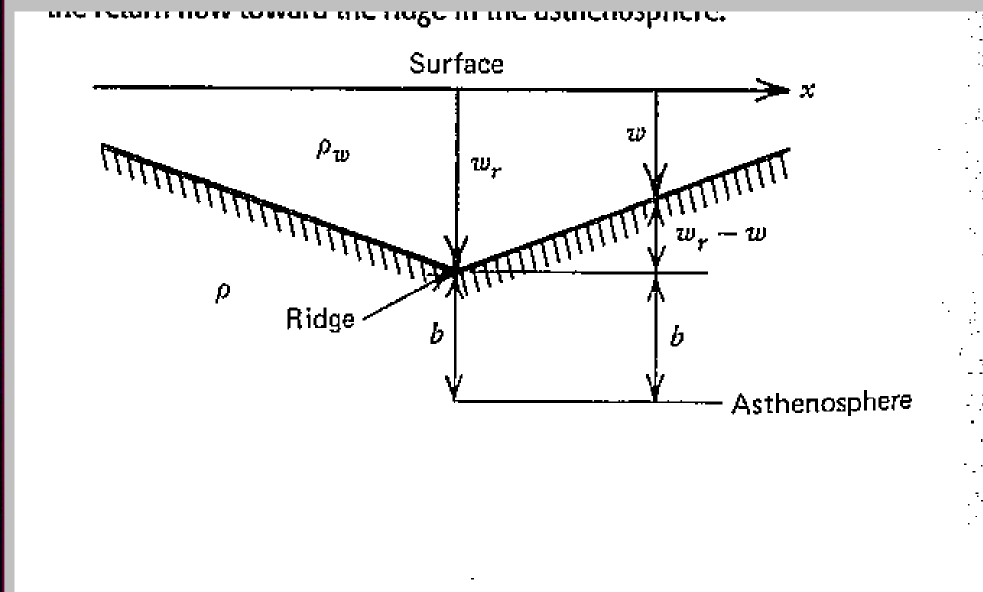


6–4 Velocity profile associated with the asthenospheric counterflow model.

But—model prediction of sea floor topography is opposite to what is observed

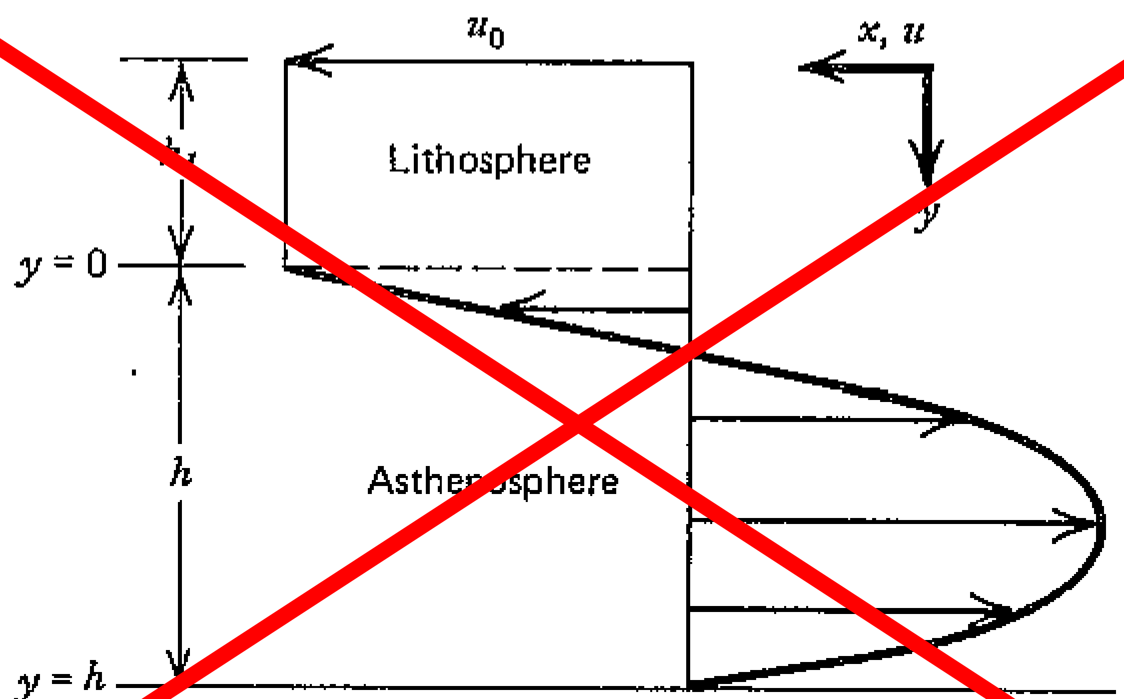
Model Prediction





Darker blue = deeper ocean
Lighter blue=shallower

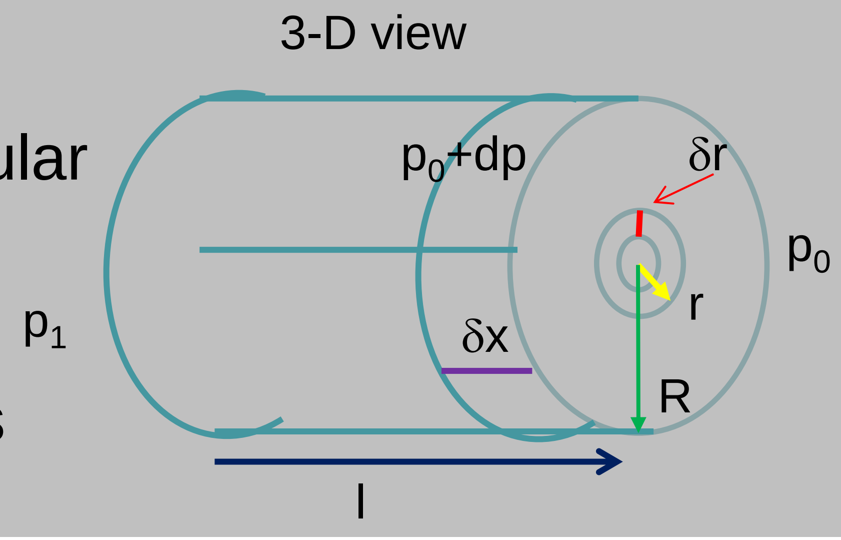
So Theory is wrong

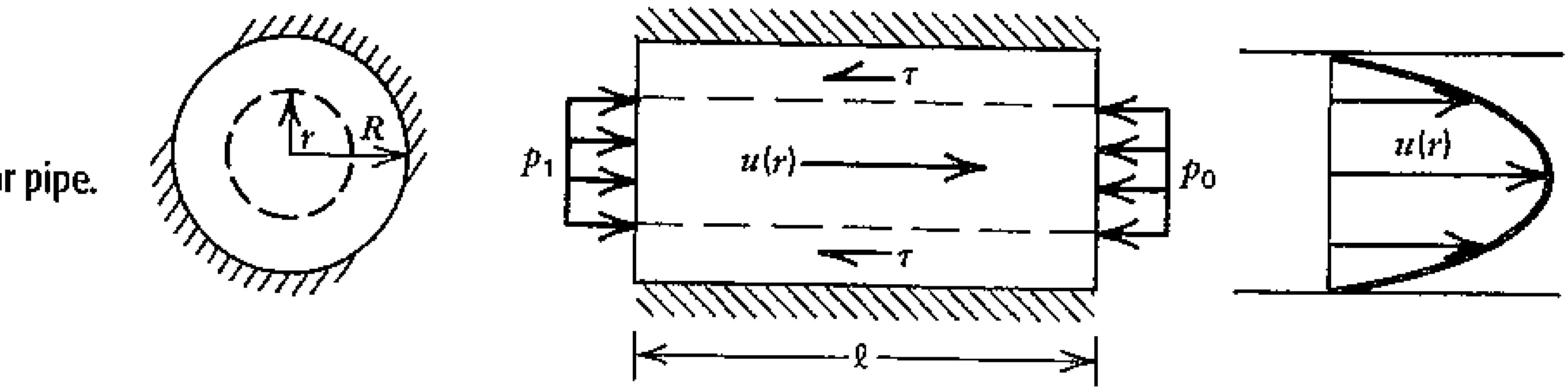


6–4 Velocity profile associated with the asthenospheric counterflow model.

6-4 Pipe Flow

- Poiseuille flow through a circular pipe
- Fig. 6-6
- Force balance works if flow is *steady*--i.e., laminar





Equations :

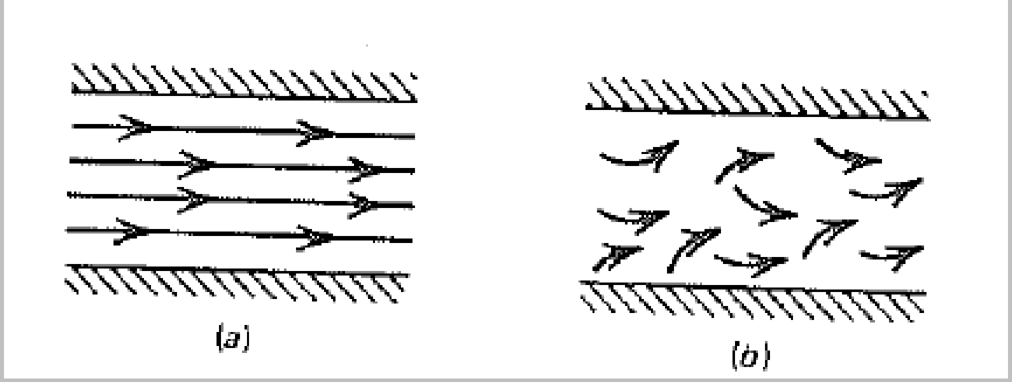
$$\tau = \frac{r}{2}\frac{dp}{dx} = \mu \frac{du}{dr}$$

Integrate to get:

$$u = -\frac{1}{4\mu}\frac{dp}{dx}(R^2 - r^2)$$

Can also calculate average velocity $\overline{u}$

But if flow is not steady, can't solve analytically



laminar turbulent

Depends on dimensionless variables: Friction factor *f* and Reynolds number *Re*

$$f \equiv \frac{-4R}{\rho \bar{u}^2} \frac{dp}{dx}$$

Reynold's Number Re: dimensionless

- D=dimension of problem (e.g., pipe diameter)
- μ=dynamic viscosity
- ν=kinematic viscosity

$\bar{u} = \text{avg speed}$

$$\text{Re} = \frac{\rho \bar{u} D}{\mu} = \frac{\bar{u} D}{\nu}$$

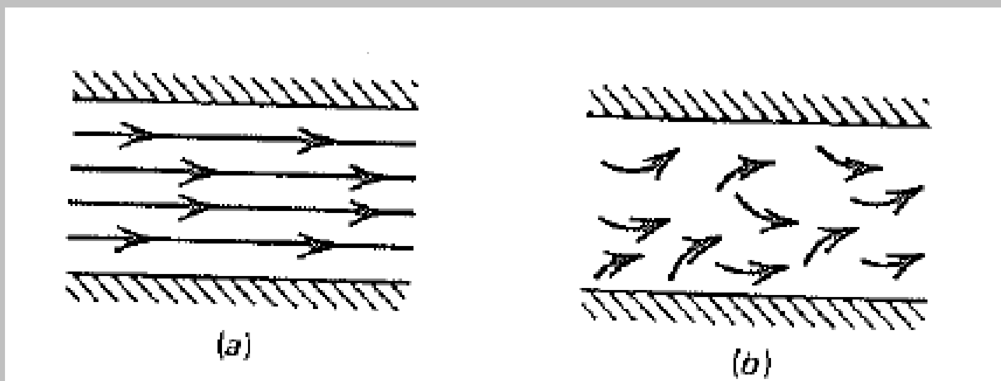
Re>2200➔turbulent flow

- Re<2200➔laminar flow
- Re<1➔ Stokes flow = reversible—movie

- http://web.mit.edu/fluids/www/Shapiro/ncfmf.html

- Low-Reynolds-Number Flows
- You-tube version has full movie on it.

But if flow is not steady, can't solve analytically

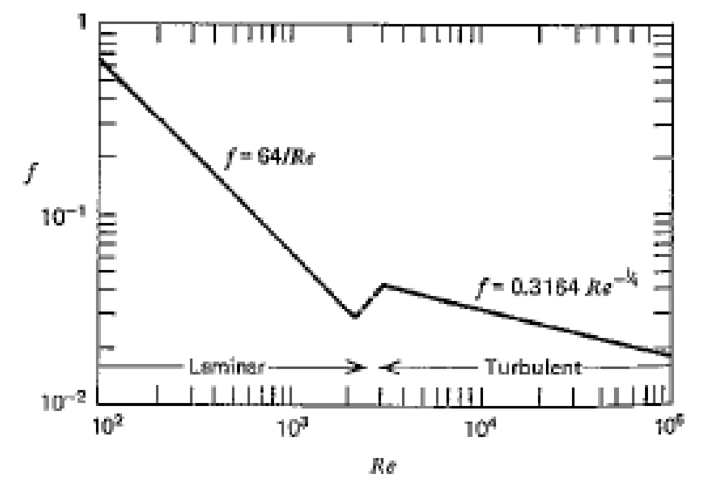


laminar turbulent

Depends on dimensionless variables: Friction factor *f* and Reynolds number *Re*

$$f \equiv \frac{-4R}{\rho \bar{u}^2} \frac{dp}{dx}$$

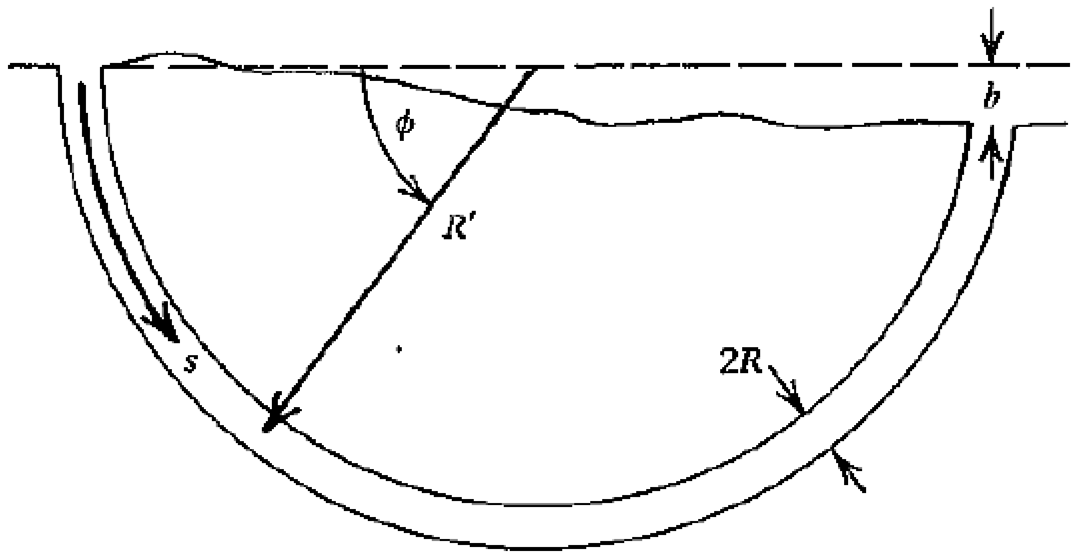
6–7 Dependence of the friction factor *f* on the Reynolds number Re for laminar flow, from Equation (6–41), and for turbulent flow, from Equation (6–42).



Example: Artesian Aquifer

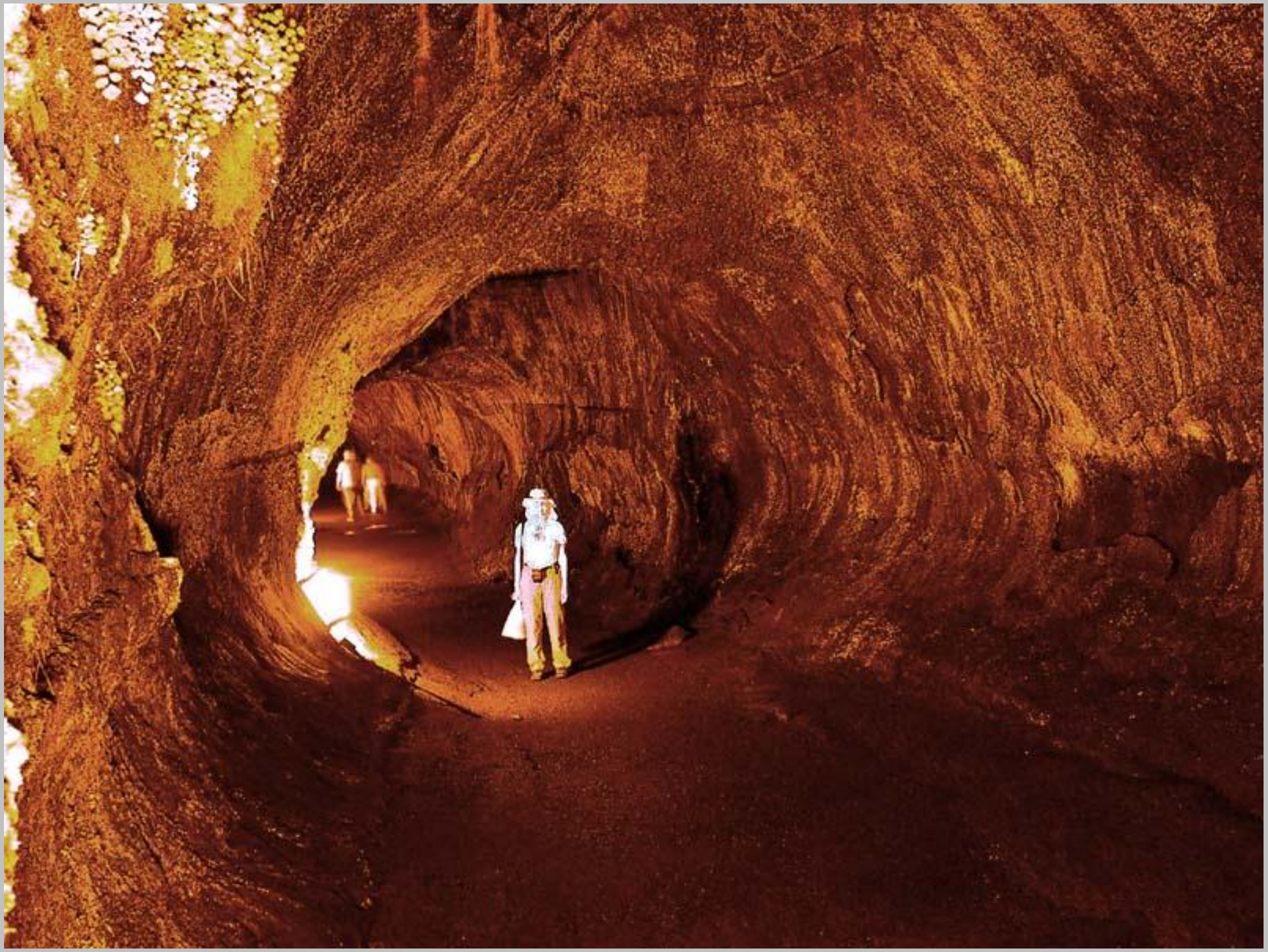
- Model acquifer basically by a pipe that is bent into a semicircle.
- Pressure difference ρgb drives flow through pipe of length $\pi R'$

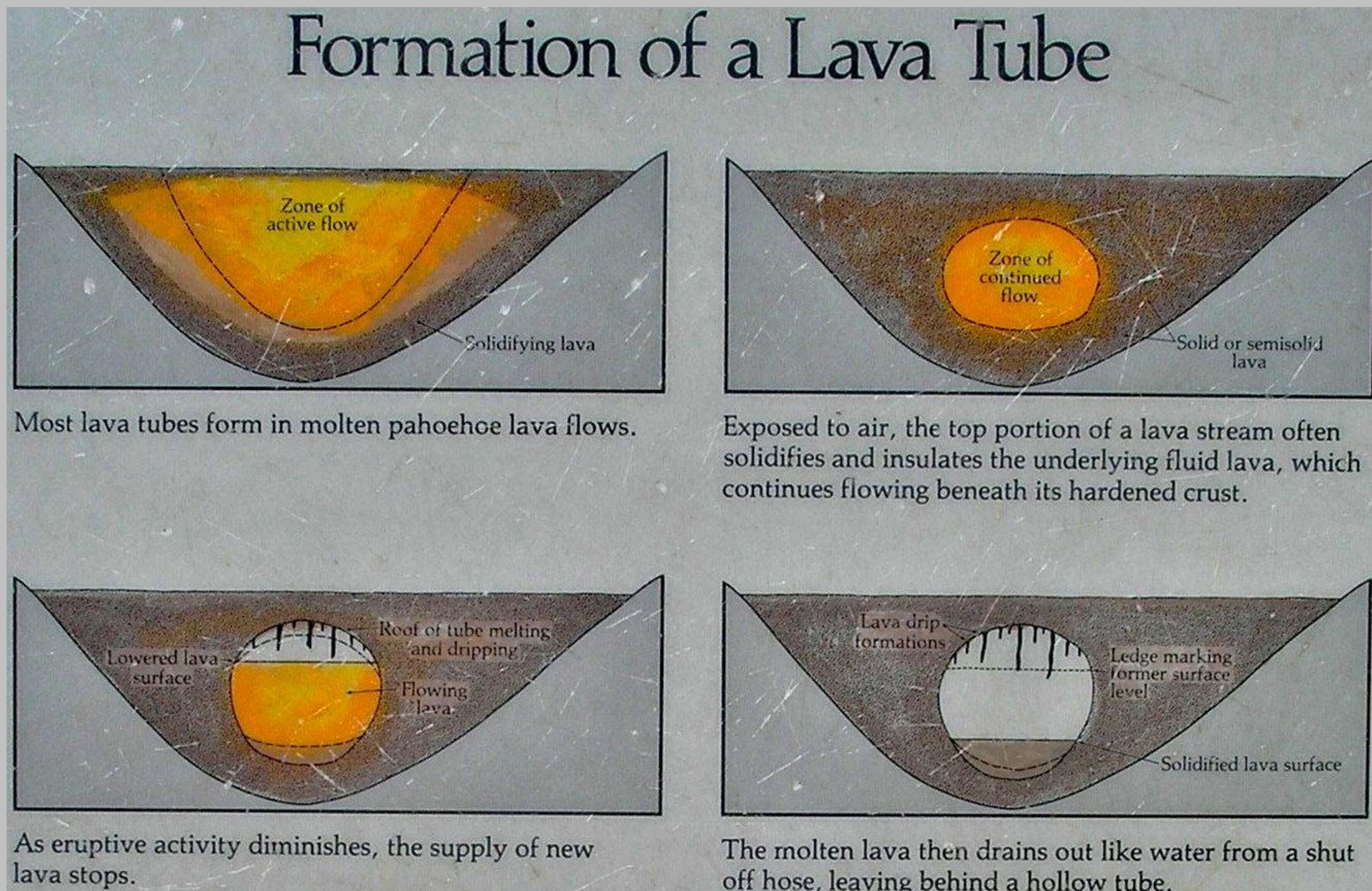
6–5 ARTESIAN AQUIFER FLOWS 233



6–9 A semicircular aquifer with a circular cross section (a toroid). A hydrostatic head *b* is available to drive the flow.

Example: Flow through volcanic pipes





http://1.bp.blogspot.com/-7UyN-xvzQYY/UbvmLdtXhsI/AAAAAAAATeM/mczCzjeMYo8/s1600/y03+Information+on+the+formation+of+lava+tube.jpg

But most lava movement is vertical—vertical magma pipe

- Driving force is buoyancy

ρ_s = density of solid

ρ_l = density of liquid

$-g(\rho_s - \rho_l)$pressure that drives magma to surface (negative for upward flow)