Notes for Assignment 2
Maths 323 fluids 2014

Last time:

Continuity Equation (Sec 6-7)

Force Balance (Sec 6-8)

Stream Function (Sec 6-9)

Postglacial Rebound (Sec 6-10)

Angle of Subduction (Sec. 6-11)

Diapirs intro (Sec 6-12)

Notes for Assignment 2
Maths 323 fluids 2014 Day 2

This time:

Diapirs (Sec 6-12)

Stokes Flow (Sec 6-14)

Equations so far

Continuity Equation in 2D

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad where$$

$$u = \frac{\partial x}{\partial t}; v = \frac{\partial y}{\partial t}$$

3-D: $\vec{\nabla} \cdot \vec{u} = 0$

Balance of pressure and viscous forces

$$\text{Eq}(6-67) \quad \frac{\partial P}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

3-D: $\vec{\nabla} P = \mu \nabla^2 \vec{u}$

$$\text{Eq}(6-68) \quad \frac{\partial P}{\partial y} = \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Where P=p-ρgy=deviatoric stress

$$u = -\frac{\partial \psi}{\partial y}; v = \frac{\partial \psi}{\partial x}$$

Stream function ψ

Biharmonic Equation: $\nabla^4 \psi = 0$

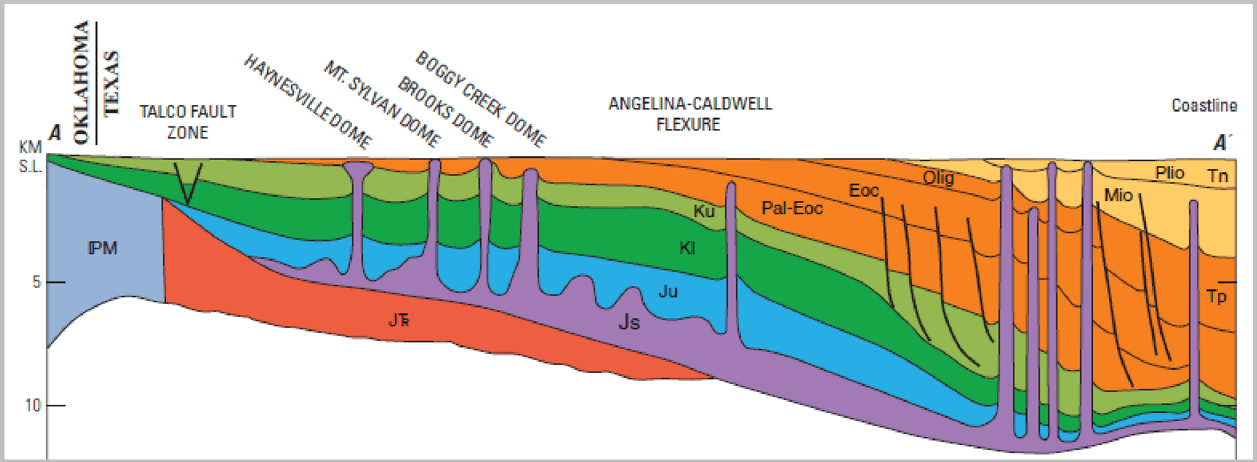
$$Q = \int_A^B d\psi = \psi_B - \psi_A$$

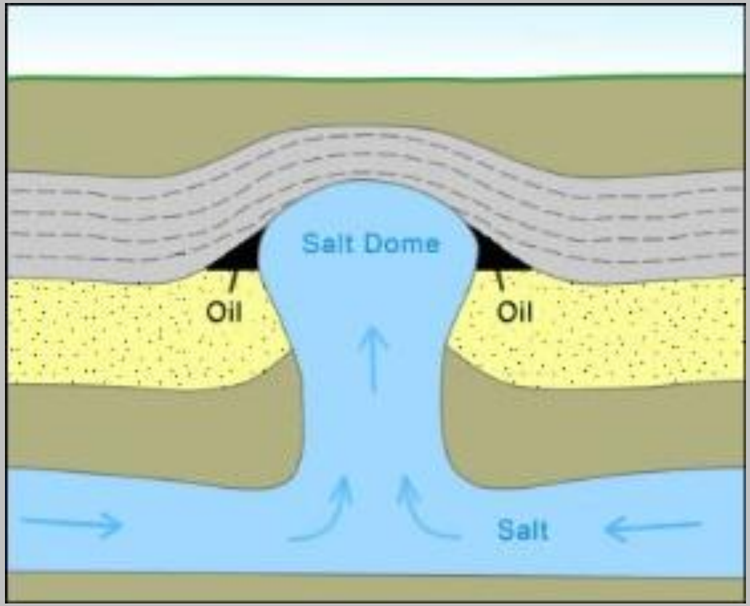
Flow rate from integral of stream function

Diapirs (Rayleigh-Taylor Instabilities) (not nappies)

- Driven by gravity and density imbalances—high over low
- Examples:
  - Paint dripping
  - Mantle “drips”
  - Start of convection, plumes, lava lamps
  - Salt domes
- Could grow exponentially until it breaks up, or could die out--returning to original state (but not periodic—not elastic)

Salt Domes





Images from : http://geology.com/stories/13/salt-domes/

Basic Eqn: Incompressible continuity Eqn

$$\vec{\nabla} \cdot \vec{u} = 0 \text{ or } \nabla^4 \psi = 0$$

Balance Buoyancy Forces by Pressure Forces:

$$\vec{\nabla} P = \vec{\nabla}(p - \rho g y)$$

P=Pressure generated by fluid flow

p=pressure

Buoyancy=ρgy

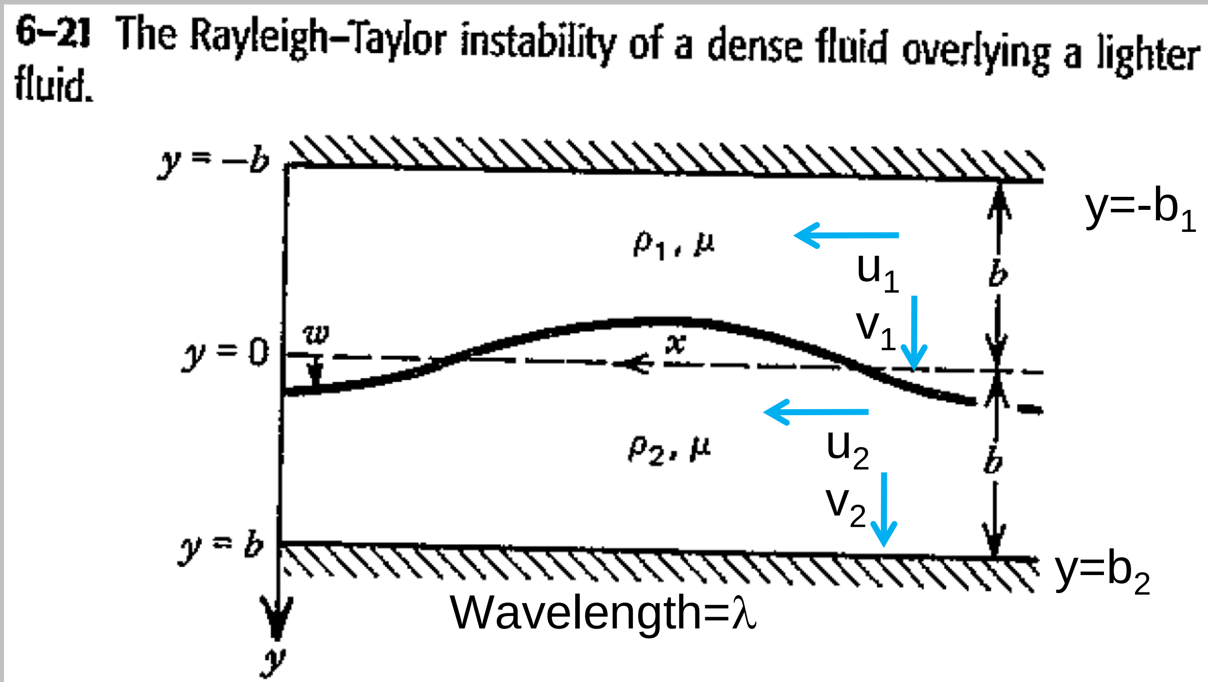
$$\vec{\nabla} P = \mu \nabla^2 u$$

(6-67 to 6-68)

=0 if forces are in balance (e.g., eqn 6-151)

To solve eqn—introduce stream function ψ

Like postglacial rebound or subducting plate—but boundary conditions differ



In general, $b_1 \neq b_2$

Displacement $w << b_1$ and b_2 -- approximation is very important –i.e., Interface shape is $w = A\cos 2\pi x/\lambda$

Because A is small, can treat interface as if it were at y=0 for the purposes of solving boundary conditions

- Boundary conditions:
  - 1)Rigid at top and bottom ($-b_1$ and b_2)—no slip condition (u continuous)

$\therefore u=v=0$ at $y= -b_1$ and b_2

  - 2) Displacements and velocities and shear stress must be continuous across boundary between media (i.e., at interface, but since w is small, effectively y=0 here)

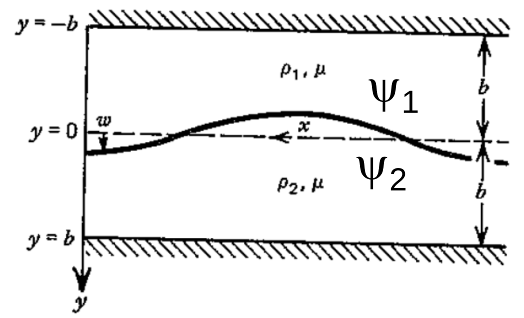
Guess solutions of ψ

- ψ_1; ψ_2 separate for each of top, bottom.
- ψ is similar in form to postglacial rebound, but uses hyperbolic functions instead of simple sines and cosines:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

6-21 The Rayleigh–Taylor instability of a dense fluid overlying a lighter fluid.

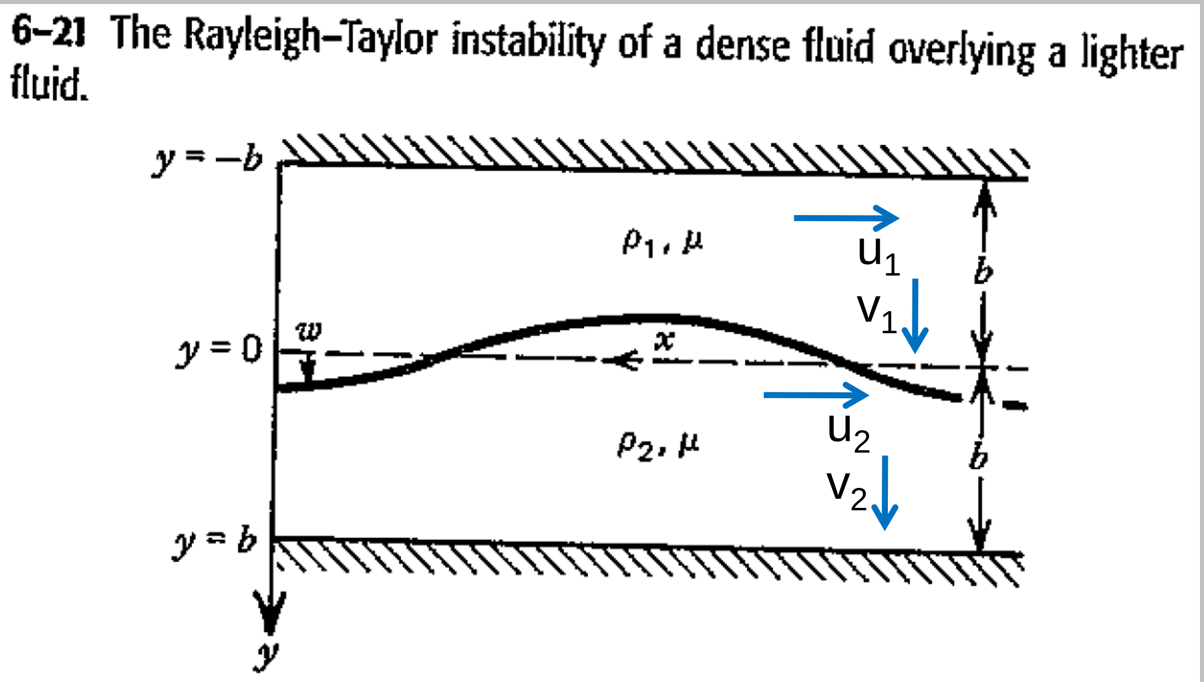


$$\psi_1 = \sin\frac{2\pi x}{\lambda}\left(A_1\cosh\frac{2\pi y}{\lambda} + B_1\sinh\frac{2\pi y}{\lambda} + C_1 y\cosh\frac{2\pi y}{\lambda} + D_1 y\sinh\frac{2\pi y}{\lambda}\right)(6-125)$$

(similar expression for ψ_2)

Solve by:

- Show that both $\psi_{1,2}$ are solns by substituting back into eqn,
- Determine $u_{1,2}$ and $v_{1,2}$ from derivatives of $\psi_{1,2}$
- $$u_{1,2} = -\frac{\partial \psi_{1,2}}{\partial y}; v_{1,2} = \frac{\partial \psi_{1,2}}{\partial x}$$
- Boundary conditions:
- u=v=0 at y= $-b_1$ and b_2 ➔ u(x,y) become $u_1(x,-b_1)=0$; $v_1(x,-b_1)=0$ $u_2(x,b_2)=0$; $v_2(x,b_2)=0$



- Boundary conditions:
  - 1)Rigid at top and bottom ($-b_1$ and b_2)—no slip condition (u continuous)

$\therefore u=v=0$ at $y= -b_1$ and b_2

  - 2) Displacements and velocities and shear stress must be continuous across boundary between media (i.e., at interface, but since w is small, effectively y=0 here)

2) velocities and shear stress must be continuous across boundary between media (i.e., at y=0 here because w is small)

- $u_1(x,0)=u_2(x,0)$; $v_1(x,0)=v_2(x,0)$

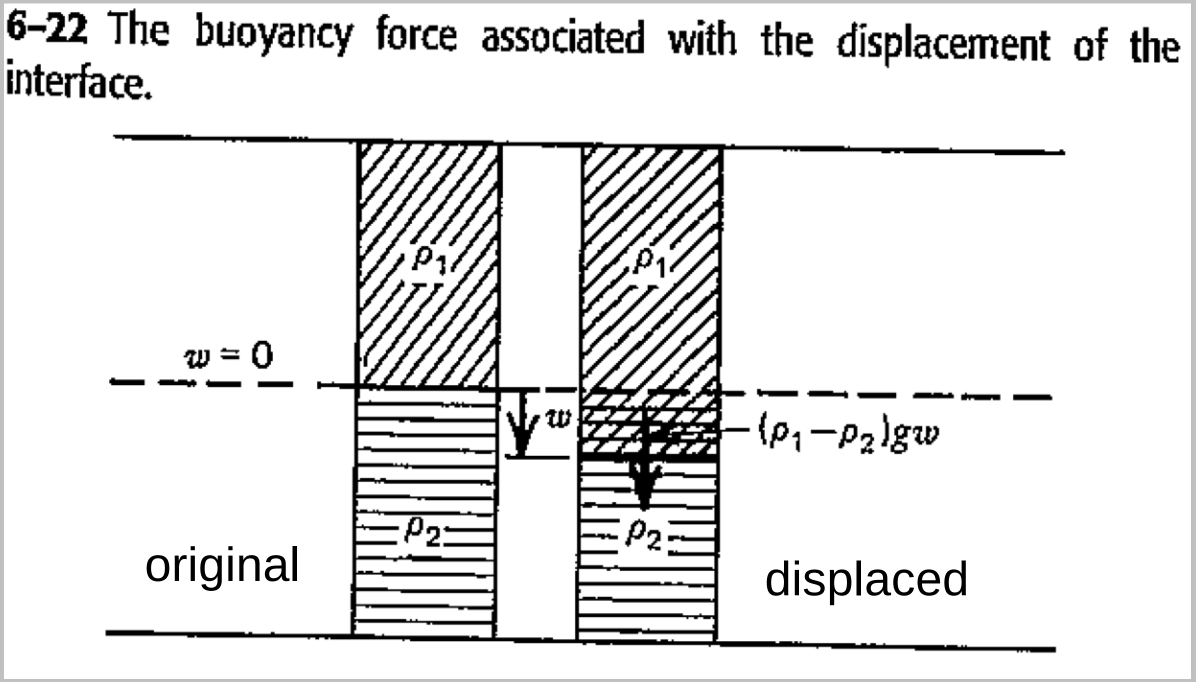
$$\tau_{xy} = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \text{ is same at boundary,}$$

$$\mu(\frac{\partial u_1(x,0)}{\partial y} + \frac{\partial v_1(x,0)}{\partial x}) = \mu(\frac{\partial u_2(x,0)}{\partial y} + \frac{\partial v_2(x,0)}{\partial x})$$

- (x dependence is purely a function of $\sin(2\pi x/\lambda)$)
- Another key—interface is moving with the same velocity as the fluid, so at y=0

$$\frac{\partial w}{\partial t} = v(x,0)$$

Finally, balance forces--buoyancy and fluid flow pressure



$$(\rho_1 - \rho_2)gw = (P_2 - P_1)\ \mathrm{at\ y} = 0$$

Buoyancy

Flow pressure found from integrating 6-72

$$\frac{\partial P}{\partial x} = -\mu\left(\frac{\partial^3 \psi}{\partial x^2 \partial y} + \frac{\partial^3 \psi}{\partial y^3}\right)$$

Final solution after much algebra:

- Solution:

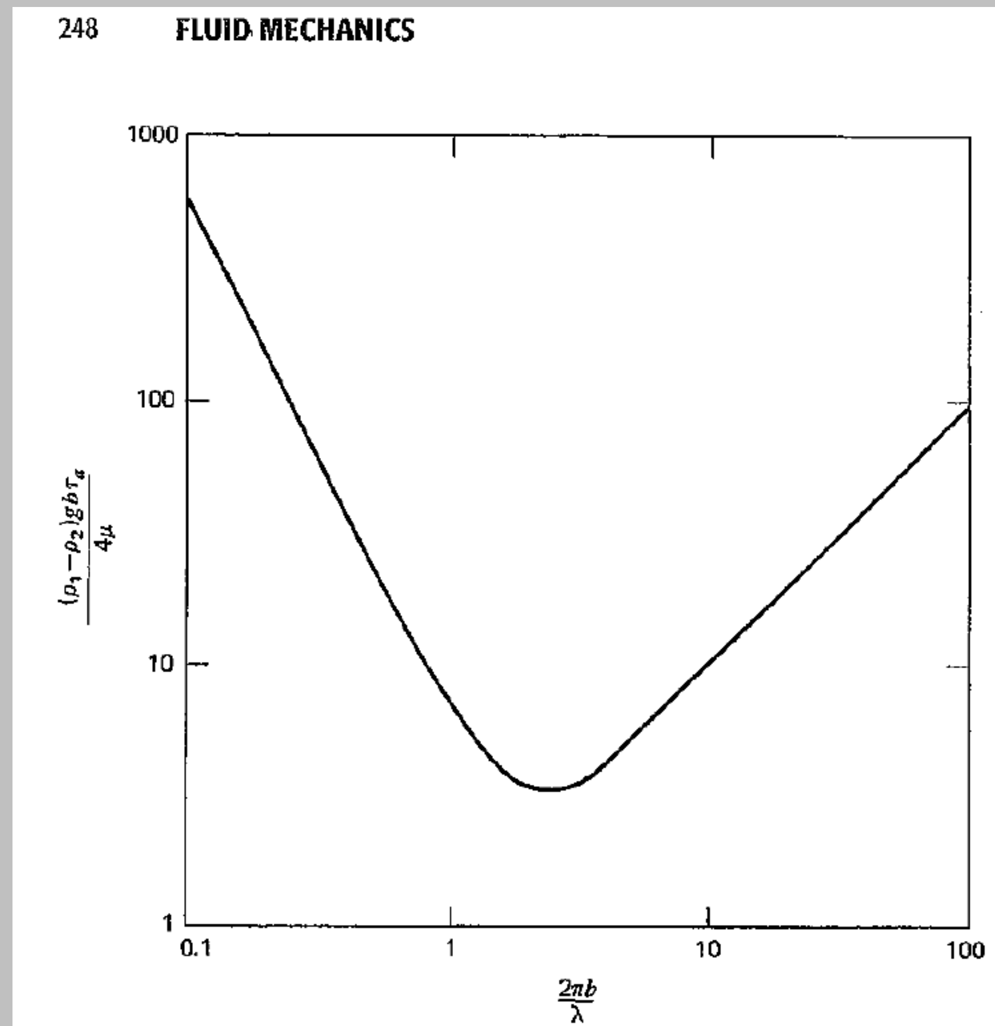
$$w = w_0 e^{t/\tau_a}$$

- Where τ_a is the growth time of the disturbance
- τ_a (Eqn 6-158) is a function of sinh, cosh($2\pi b/\lambda$) multiplied by

$$\frac{4\mu}{(\rho_2 - \rho_1)gb}$$

- τ_a depends on wavelength, but if have displacements at multiple wavelengths, fastest growing wavelength will dominate (τ_a is a minimum)

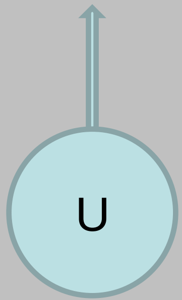
Dimensionless growth time of disturbance



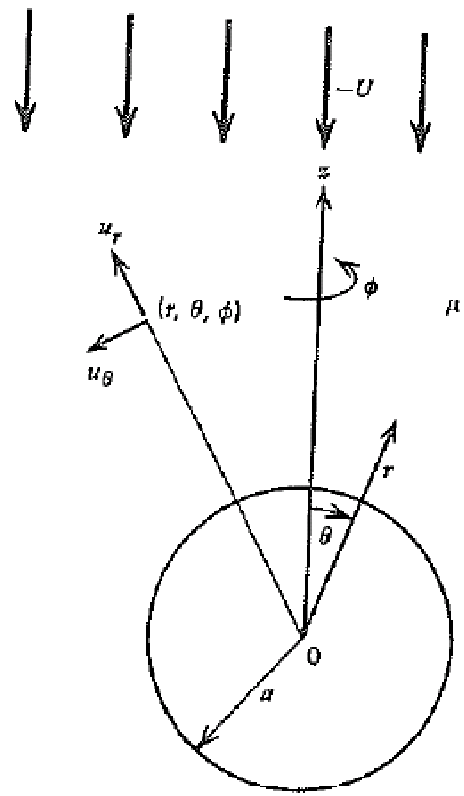
Dimensionless wavenumber

Stokes' Flow: How fast does a body fall due to its own weight?

- Applies in limit of very viscous fluid, with Re<1 (reversible flow)
- Applications:
  - Fall of pieces of slab
  - Rise of plumes/magma
  - Fall of metal probe

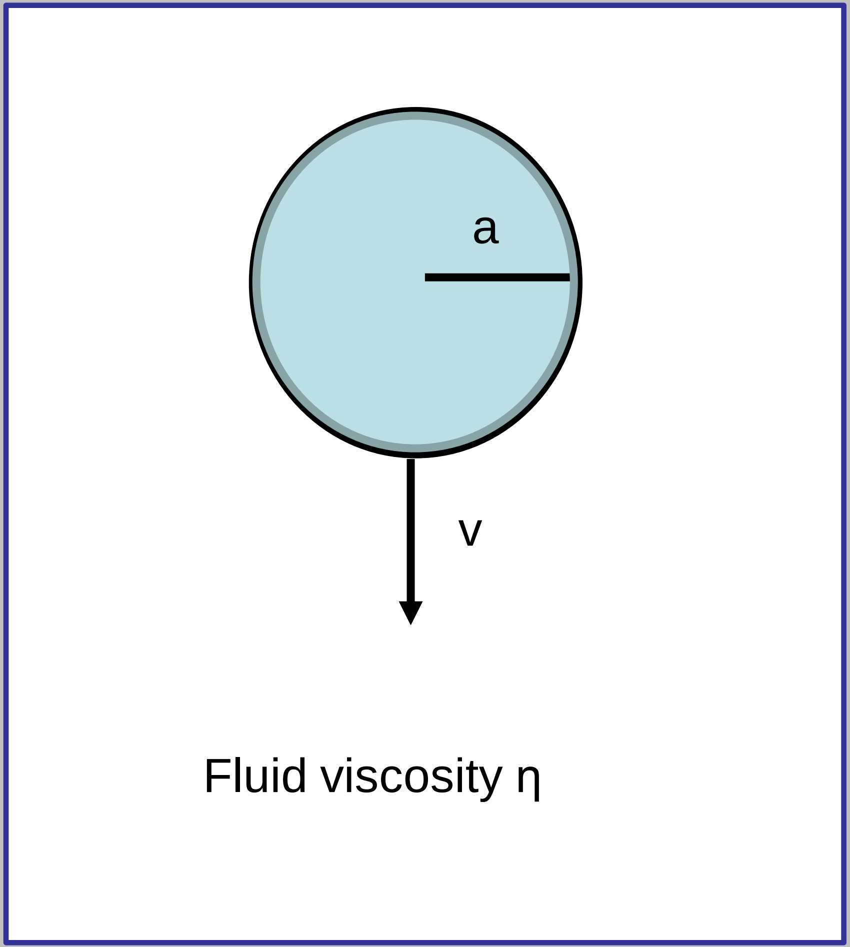


Ball rises through stationary fluid or fluid flows past stationary ball

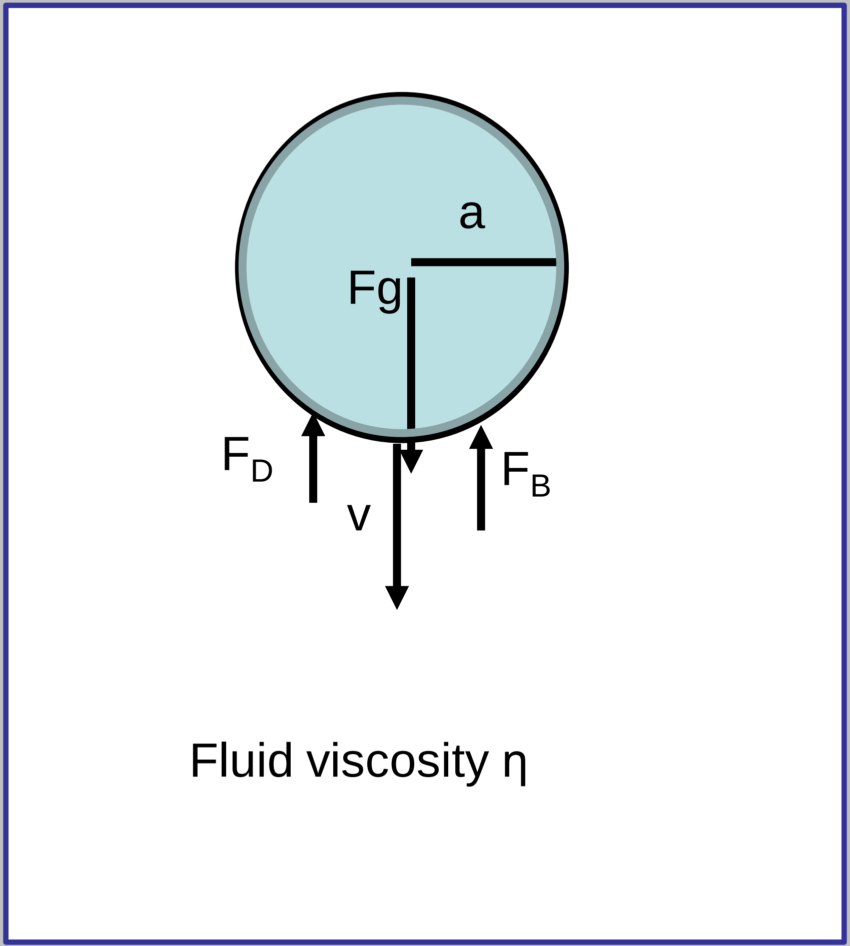


6–31 Steady flow of a viscous fluid past a sphere.

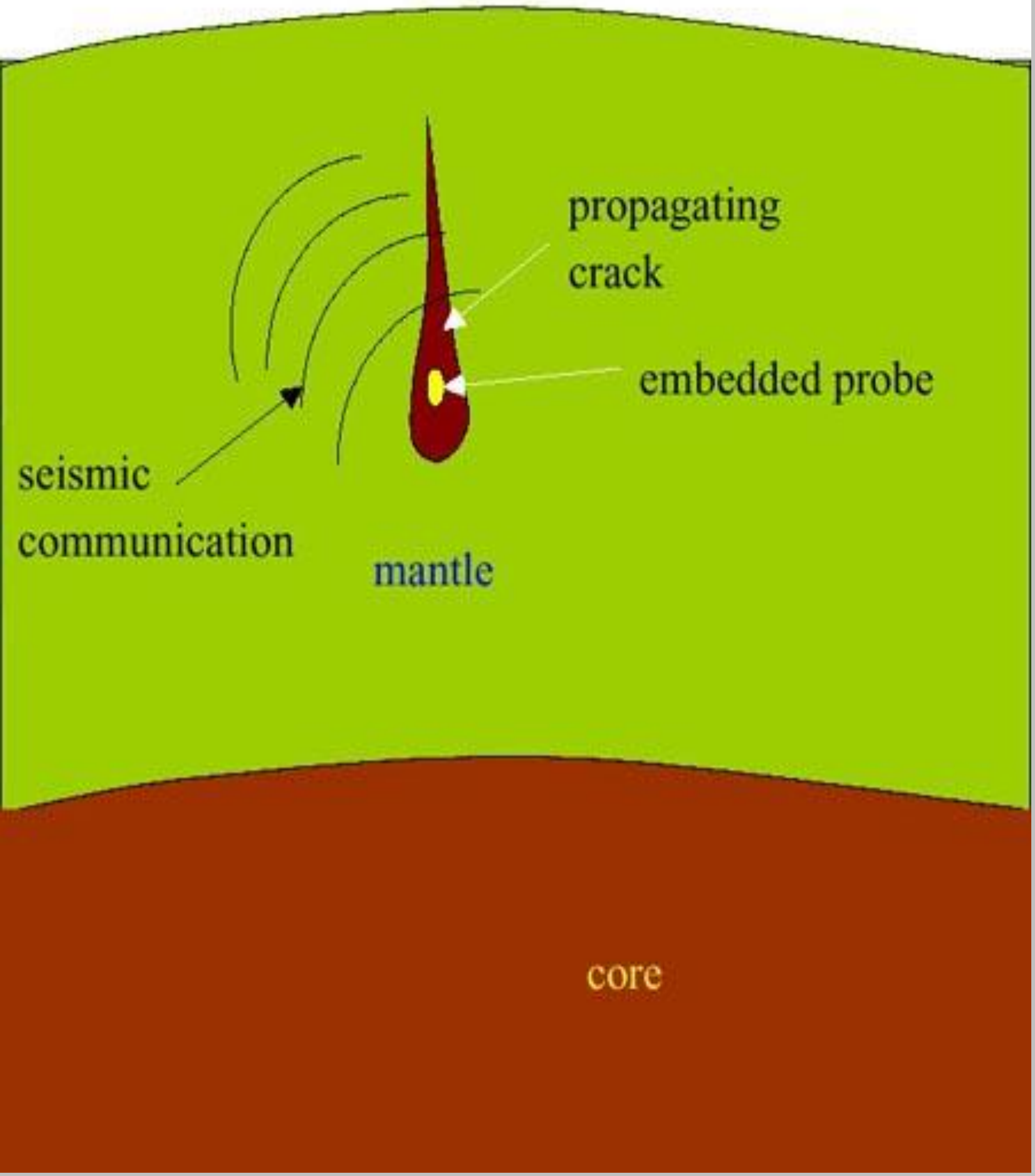
Sphere Falling in a Fluid



Sphere Falling in a Fluid



$F_g + F_B + F_D = 0$



Fall of Iron into Core

Stevenson, David J. Mission to Earth's Core -A Modest Proposal. Nature, 423, 239-240, 2003. (in course notes)

About 1 week to get to core

Balance gravity (Buoyancy) and Viscous drag forces

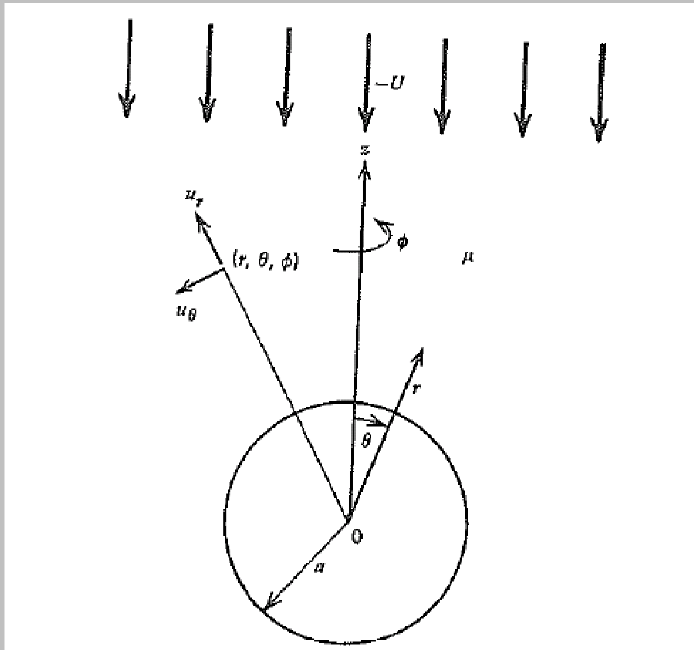
- Dominant equations: continuity equation and pressure equation again, same as before but now geometry and boundary conditions change

$$\vec{\nabla}\cdot\vec{u}=0 \qquad \vec{\nabla}P=\mu\nabla^2 u$$

- Where P=p-ρgy

- ρ_f=density of fluid
- ρ_s=denisty of sphere

$$\mathrm{Re}=\frac{\rho_f U(2a)}{\mu}$$



6–31 Steady flow of a viscous fluid past a sphere.

Boundary Conditions

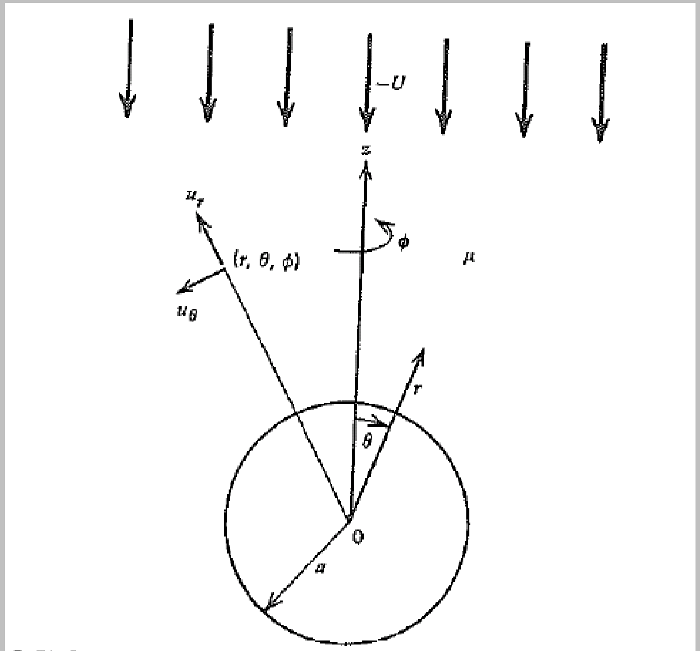
- As $r \rightarrow \infty$

  $u_r \rightarrow -U$ in z direction

  $u_r \rightarrow -U\cos\theta \quad u_\theta \rightarrow U\sin\theta$

No-slip on sphere: at r=a

$u_r = u_\theta = 0$



6-31 Steady flow of a viscous fluid past a sphere.

Spherical Coordinates:

Continuity equation becomes:

$$0 = \vec{\nabla} \cdot \vec{u} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (u_\theta \sin\theta) + (\frac{1}{r \sin\theta} \frac{\partial u_\phi}{\partial \phi})$$

But since u_Φ=0, last term is 0

To solve equation, also need the Laplacian of u:

$$\nabla^2 \vec{u} = \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{u})$$

$$\vec{\nabla} \times \vec{u} = \frac{1}{r \sin\theta} [\frac{\partial}{\partial \theta} (u_\phi \sin\theta) - \frac{\partial u_\theta}{\partial \phi}] \hat{r} + \frac{1}{r} [\frac{1}{\sin\theta} \frac{\partial u_r}{\partial \phi} - \frac{\partial (r u_\theta)}{\partial r}] \hat{\theta} + \frac{1}{r} [\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}] \hat{\phi}$$

Pressure forces: Terms in P

Viscous forces: Terms in $\mu \nabla^2 u$

Solution

- Surprisingly, most terms drop out and …
- Pressure due to fluid flow is (Eq 6-216):

$$p = \frac{3\mu a U}{2r^2}\cos\theta$$

- Integrate to get downward “drag” (force) due to fluid pressure across sphere: $D_p = 2\pi\mu a U$

Viscous drag:

- Using 3-D formulation of stress again:

$$\overleftrightarrow{\tau} = \mu(\vec{\nabla}\vec{u} + \vec{\nabla}\vec{u}^T)$$

Integrate to get Viscous Drag $D_v = 4\pi\mu aU$

So total Drag F_D= Viscous Drag + Pressure Drag = $D_p + D_v = 6\pi\mu aU$

Speed of rise or fall:

- Balance Buoyancy Forces with Drag forces for steady-state case (no acceleration):
- $F_B=(\rho_f - \rho_s)g4\pi a^3/3= F_D = 6\pi\mu aU$
- Solve for U
- For faster flow, Re>1, more difficult: use dimensionless drag coefficient C_D

$$C_D \equiv \frac{F_D}{\frac{1}{2}\rho_f U^2 \pi a^2} = \frac{24}{\mathrm{Re}} (6-226)$$

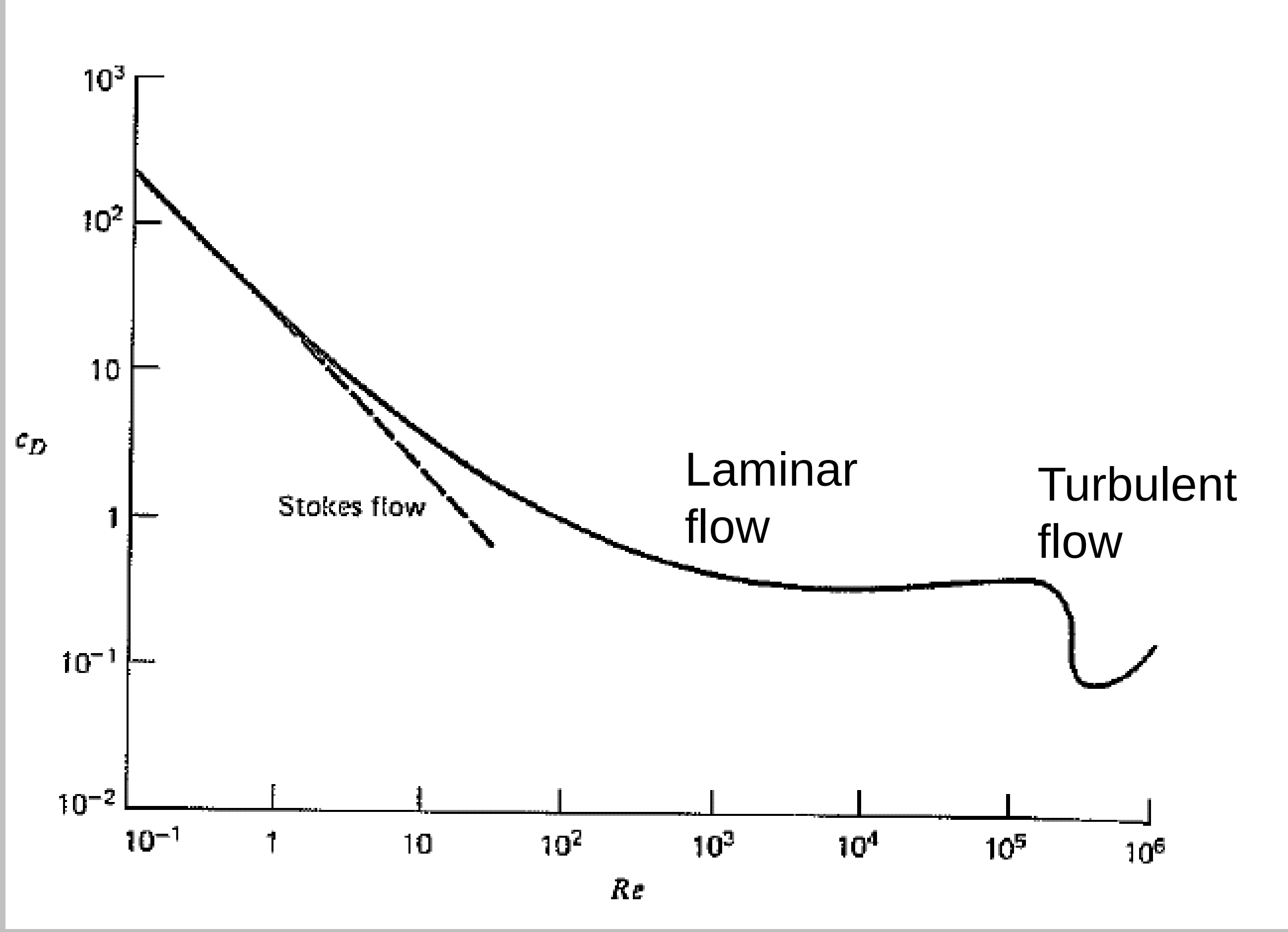
Pressure due to vel.

Sphere x-sec area (shadow)

- Stokes flow:

$$U = \frac{2(\rho_f - \rho_s)ga^2}{9\mu} (6-229)$$

- Re>1:



$$U = \left[\frac{8(\rho_f - \rho_s)ga}{3C_D\rho_f}\right]^{1/2} (6-230)$$

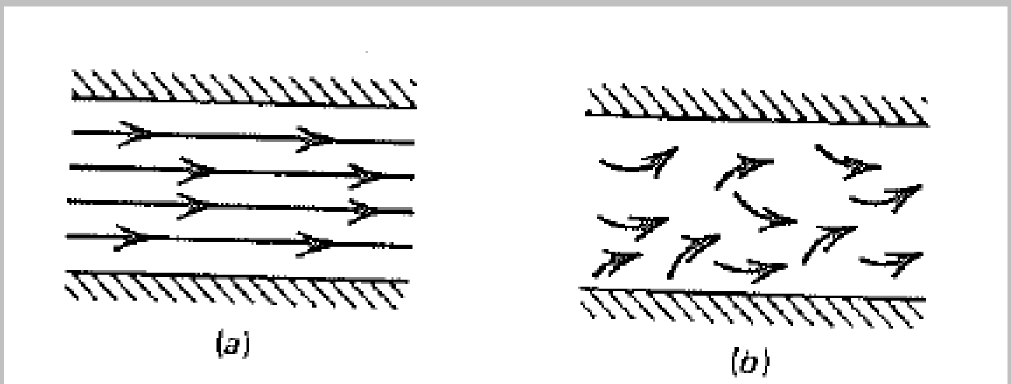
$$C_D \equiv \frac{F_D}{\frac{1}{2}\rho_f U^2 \pi a^2} = \frac{24}{\mathrm{Re}} (6-226)$$

$$\bar{u} = \text{avg speed}$$

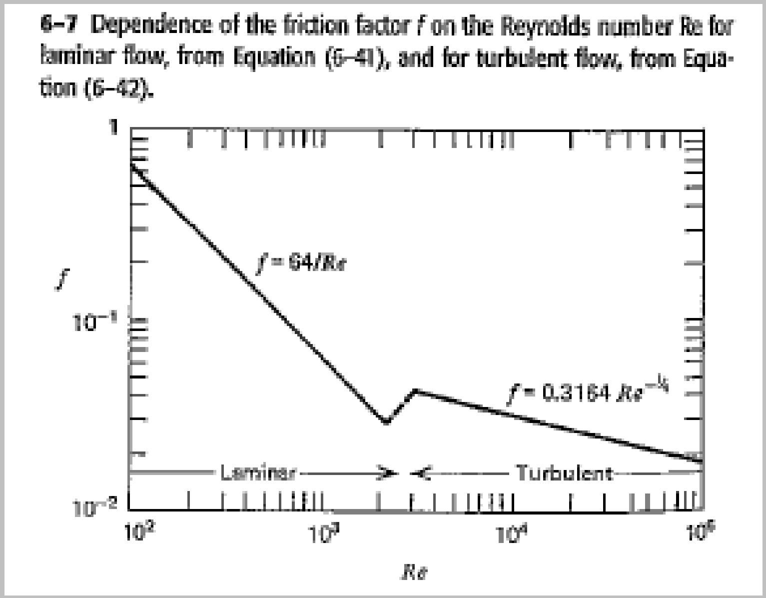
$$\mathrm{Re} = \frac{\rho \bar{u} D}{\mu} = \frac{\bar{u} D}{\nu}$$

Note—units work out in both cases

Compare to pipe flow:



laminar turbulent



Depends on dimensionless variables: Friction factor *f* and Reynolds number *Re*

$$f \equiv \frac{-4R}{\rho \bar{u}^2} \frac{dp}{dx}$$