MATH 322/323 Module 1 Cartesian Tensors Mar 5 – April 24 2012

Timetable

Week	1	2	3	4	5	6
Start	Mar 4	Mar 11	Mar 18 Assignment 1 due	Mar 25 Assignment 2 due	Apr 8 Assignment 3 due	Apr 15
Mon 1200-1250		L3	L5	L7	L9	L11
Tues 1200-1250	L1 Assignment 1 set	L4 Assignment 2 set	L6 Assignment 3 set	L8 Assignment 4 set	L10	T6
Weds 1000-1050	L2	Spare	Spare	Spare	Spare	Spare
Tutorial Fri 1200-1250	T1	T2	T3	T4	T5	Spare Assignment 4 due

Assignments and tutorial exercises

All assignments due 5pm on day of week shown.

Essay due 5pm Monday 29 April

Assessment Summary

Assignment 1	20%	Index notation; Rotational transformations; Euler vector
Assignment 2	20%	Prove Kronecker is a tensor; lead rubber bearing, stress force across a plane
Assignment 3	20%	Strain gauges – principal axes, simple shear
Assignment 4	20%	Hooke's Law, tensor calculus
Essay	20%	**due 29 April.**

Tutorial One 8 March

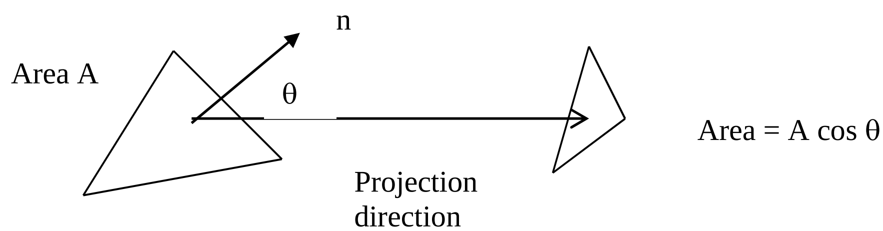
Revision – vectors and linear algebra

1. Let **a** be the position vector of a given point $(x_{10}, x_{20}, x_{30})^T$ and **r** be the position vector of any point $(x_1, x_2, x_3)^T$. Describe the locus of **r** if:

a: $|\mathbf{r} - \mathbf{a}| = 3$; b: $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{a} = 0$; c: $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{r} = 0$

2. a: Show that the area of a triangle formed by two vectors **a** and **b** is ½ | **a** x **b**|

b: Hence show that he projected area of a triangle = (area of the triangle) x cosine of the angle between the normal to the triangle and the projection direction; ie:



NB this is a very well known result, but one that is hardly ever proved!

3. If $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ are distinct vectors, construct a RH Cartesian set of axes where x_1 is normal to the plane of $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ and x_2 and x_3 are any two vectors in the plane of $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$.

4. Attempt to find a non-trivial solution to Ax = 0 **-**

(i) For A =

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 2 \\ -1 & -5 & -3 \end{bmatrix}$$

(ii) for A =

$$\begin{bmatrix} 3 & 4 & 0 \\ 2 & -1 & 0 \\ 0 & 11 & 1 \end{bmatrix}$$

Changing axes

5. $\mathbf{u_1}$ and $\mathbf{u_2}$ are two mutually perpendicular vectors, which are used to construct a new coordinate system. **y** is a vector described in the old coordinate system. What are the coordinates of the endpoint of **y** in the new system?

6. Construct transformation matrices A for giving the coordinates of a vector $\underline{\mathbf{p}}$ in a new coordinate system, using the convention
$\underline{\mathbf{p}}$ (new) = A^T $\underline{\mathbf{p}}$ (old), for:

(a) Rotation through 90^o about **x2** axis,

(b) Rotation through 45^o about **x2**, followed by rotation through 45^o about (new) **x1,**

In each case:

(i) verify that your transformation works by applying it to a suitable test vector e.g. one of the coordinate axes.

(ii) find $|A|$ (determinant A).

MATH 322/323 Cartesian Tensors

Assignment 1 ***due Monday 18 March*****.**

(1) *Ascertain whether the following are valid index set equations*. If any are invalid, re-write the RHS or the LHS to make them valid.

(a) $a_{ij} = b_{ij} c_{jk}$

(b) $a_{ijj} = b_{ik} c_k$

(c) $a_i = b_{ij} c_{jk} b_{ik}$

(2) $\underline{\mathbf{w}} \in R_k$, $\underline{\mathbf{v}} \in R_n$ and A, X, and Y are (real) matrices: A is n by k, Y is k by p, Z is k by p, and X is n by p . ^{T} denotes transpose. *Write valid index set equations for*:

(a) $\underline{\mathbf{w}}^T = \underline{\mathbf{v}}^T A$

(b) $X = A Y$

(c) $Z = A^T X$

(3) *Write in matrix notation*:

(a) $a_{ik} = b_{ij} c_{jn} d_{nk}$

(b) $a_{ik} = b_{ij} c_{nj} d_{nk}$

(c) $a_{ik} = d_{nk} c_{jn} b_{ij}$

(4) Construct transformation matrices A, using the convention
$\underline{\mathbf{p}}$ (new) = A^T $\underline{\mathbf{p}}$ (old), for:

(a) Rotation through 180^o about **x3** axis,
(b) Rotation through 45^o about **x1** axis,
(c) Rotation through θ about the **x2** axis,
(d) Rotation through 90^o about **x1**, followed by rotation through 45^o about (new) **x2,**
(e) Reflection in the **x1, x3** plane.

In each case:
(i) verify that your transformation works by applying it to a suitable test vector e.g. one of the coordinate axes.
(ii) find $|A|$ (determinant A).

Method: write down the unit vectors describing the new axes, and use the lecture results to construct A.
Hint: sketch the old and new axes for each rotation.

(2) Euler's theorem

A rigid body has a Cartesian coordinate system embedded into it. The (unit) vectors describing the axes are:

α_1	α_2	α_3
0.7229	0.5883	0.3623
-0.5623	0.1961	0.8034
0.4016	-0.7845	0.4726

It rotates about the origin to a location where the axes are now described by the vectors:

β_1	β_2	β_3
-0.7071	0.3015	0.6396
0.7071	0.3015	0.6396
0	0.9046	-0.4264

(a) Solve Chapter 1 equations (4) (for i = 1, 2) in the proof of Euler's theorem to find the axis (unit) vector $\underline{\mathbf{x}}_{\mathbf{E}}$ for the rotation. *Verify* that your vector satisfies eqn (4-3).

Hint: Since only two components of $\underline{\mathbf{x}}_{\mathbf{E}}$ are independent, solve for the ratios

$\underline{\mathbf{x}}_{\mathbf{E}\,1} / \underline{\mathbf{x}}_{\mathbf{E}\,3}$ and $\underline{\mathbf{x}}_{\mathbf{E}\,2} / \underline{\mathbf{x}}_{\mathbf{E}\,3}$.

(b) Find the angle through which the rigid body was rotated, as follows:

(i) Find (any) unit vector $\underline{\mathbf{r}}$ at right angles to $\underline{\mathbf{x}}_{\mathbf{E}}$.
(ii) Use the equations given in lectures for transforming between coordinate systems to find the description of $\underline{\mathbf{r}}$ in the α and β systems, = $\underline{\mathbf{r}}^{\alpha}$ and $\underline{\mathbf{r}}^{\beta}$.
(ii) The scalar product of $\underline{\mathbf{r}}^{\alpha}$ and $\underline{\mathbf{r}}^{\beta}$ gives the cosine of the rotation angle.

You must use a programming language (Maple, MatLab) or Excel (or similar freeware) for the arithmetic.

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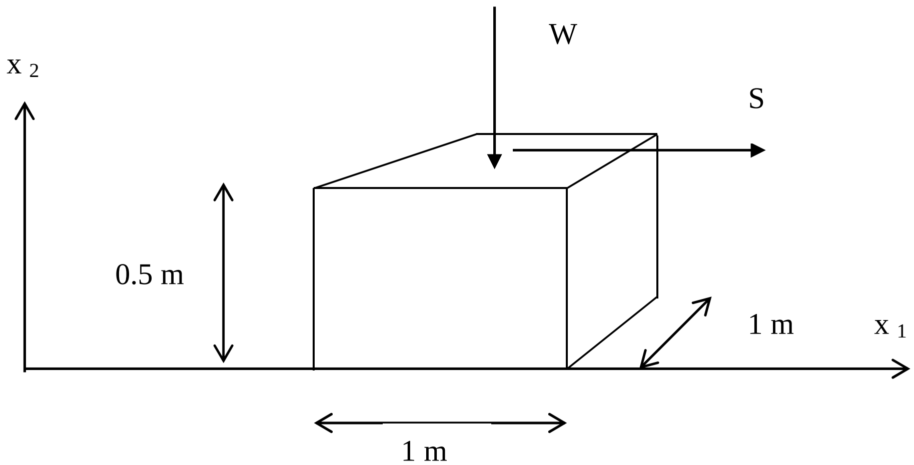
Assignment 2 *due 25 March;* **Notes Chapter 1**

(1) Show formally that the Kronecker Delta δ_{ij} is a tensor; i.e. for any (orthogonal) transformation of the coordinate system given by a_{pq} , show that δ_{ij} satisfies:

$$\delta'_{ij} = a_{ip} a_{jq} \delta_{pq}$$

(2) Show formally that the index set defined by $x_j = 1$, $j = 1,2,3$ for *every* set of Cartesian coordinate axes, is *not* a tensor. NB if it fails the test for any one transformation, it is not a tensor.

(3) Lead rubber bearings for damping earthquake motions have been fitted to the columns of the Rankine Brown building and Te Papa. They are tested by applying a load W equal to the share of the weight of the building and then applying a shear force S to simulate earthquake forces:

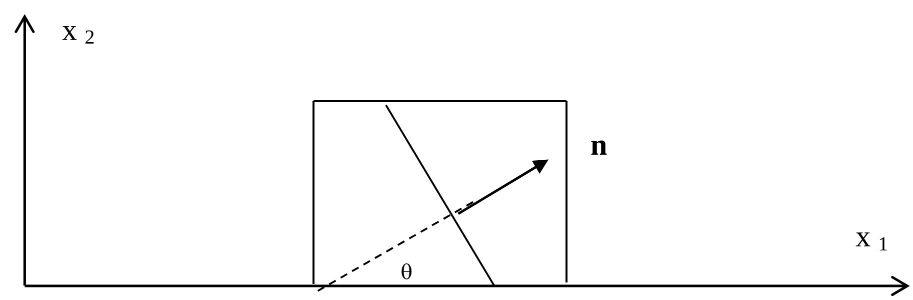


A lead rubber bearing is modelled as a homogeneous cuboid 1m x 1m bearing area by 0.5 m high. If W = 50 MN and S = 10 MN,

(i) What additional forces must be applied to keep the block in equilibrium? (i.e. stop the block rotating or accelerating)?

(ii) Write down the stress tensor for the bearing.

(iii) Find the Principal Axes of the stress tensor and the Principal Stresses.



(iv) Find the stress force F per unit area inside the block across a plane with its normal in the $x_1 x_2$ plane, making an angle θ with the x_1 axis i.e. $\mathbf{n} = (\cos\theta, \sin\theta, 0)$.

(v) Write down expressions for the Normal, N, and total Shear, S_T , components of F, and find the orientation(s) of the plane which makes the ***magnitude*** of each, separately, a maximum.

(vi) Hence find the maximum Shear and Normal stresses in the block.

Hints:

1. To find the shear force, find the direction of the total Shear force, and take the scalar product with F.
2. Write the expressions for $|N|$ and $|S_T|$ in terms of 2θ before differentiating to find the maxima.

Tutorial Two 15 March AND Tutorial Three 22 March

(0) Complete any questions from Tutorial one.

(1) Construct transformation matrices A for giving the coordinates of a vector $\underline{\mathbf{p}}$ in a new coordinate system, using the convention
$\underline{\mathbf{p}}$ (new) = A^T $\underline{\mathbf{p}}$ (old), for:

(a) Rotation through θ^o about **x1** axis,
(b) Rotation through θ^o about **x2** axis,
(c) Rotation through θ^o about **x3** axis

2. Show formally that the Alternating Tensor ε_{ijk} is a tensor; i.e. for any (orthogonal) transformation of the coordinate system given by a_{pq}, show that ε_{ijk} satisfies:

$$\varepsilon'_{ijk} = a_{ip} a_{jq} a_{kr} \varepsilon_{pqr}$$

3. If a continuum is subject to a stress S_{ij} at a point P, find expressions for the Normal and total Shear components of force across any plane through P.

4. S is given by

$$S = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Find the Normal N and total Shear force S components across a plane with normal
$\underline{\mathbf{n}}^T$ = $(\cos\theta , \sin\theta , 0)$.

Hence show that the pair of values (N, S) lie on a circle in the N, S plane centred at $\{(S_1 + S_2)/2 , 0\}$ with radius $| S_1 - S_2 |/2$ (This is called the Mohr Circle). Hence find the magnitudes of the maximum Normal and Shear stresses, and the directions they act in.

5. If **F** is the stress force exerted across a plane P, show that the stress force exerted across any plane that contains **F** lies in the plane of P.

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Assignment 3 ***due 8 April*.**

(1) a, b, c are three strain gauges 60° apart (c.f. Q1 in the tutorial exercises, where a, b, c are 45° apart).

If ε_a , ε_b , ε_c , are the strains measured in the directions a, b, c respectively, find the strain tensor, the Principal Strains and the directions of the Principal Axes.

Hint. Solve the quadratic equation for the eigenvalues using tensor components of strain before you substitute for ε_a , ε_b , ε_c . Do not spend a lot of time trying to simplify the resulting expression.

(2) A continuum deforms as follows: the displacement Δu_i of any point P relative to the origin is of the form:

$$\Delta u_i = (u_1, 0, 0)^T ,$$

where $u_1 = k\, a_2$, where k is a constant << 1, and a_1, a_2, a_3 are Lagrangian coordinates (this deformation is called *Simple Shear*).

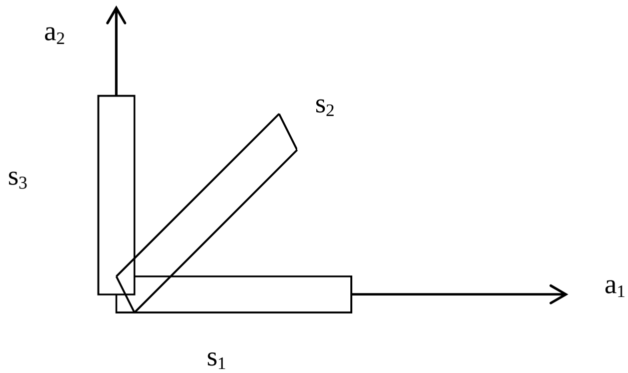
Find the Strain and Rotation tensors E and W, the equivalent rotation vector $\underline{\boldsymbol{\omega}}$, and the Principal Strains and Principal Axes of E. What is the dilatation?

(3) Find the Principal Axes, Principal Strains and dilatation for a continuum where the strain tensor is:

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix} \times 10^{-4}$$

Tutorial Four 29 March

(1) We have three strain gauges (ie instruments which measure the change in a length of wire, or the distance between two points using a laser, etc) deployed 45^o apart in a plane, as shown.



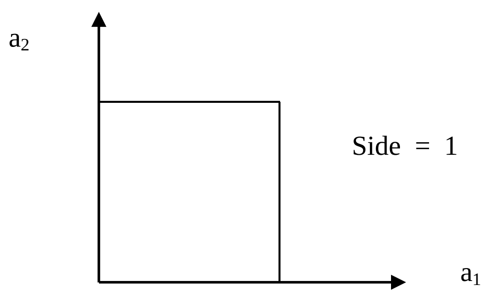
The material the strain gauges are mounted on suffers a strain E and each registers a (linear) strain = ε_1, ε_2, ε_3 respectively. That is, there is a change of :

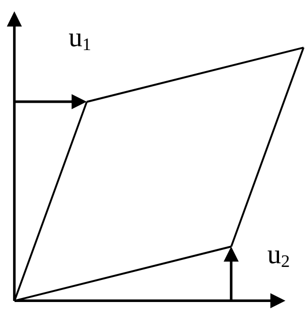
- ε_1 per unit length in the direction of s_1, which is a_1;
- ε_2 per unit length in the direction of s_2, which is at 45^o to a_1;
- ε_3 per unit length in the direction of s_3, which is a_3;

Without loss of generality, we can take each strain gauge to have unit length. For each of s_1, s_2, s_3 take their end-points to be on a unit circle.

Calculate the strain tensor E using strain gauge measurements.

(2) Consider a unit square: which deforms to:





There is no deformation, or change, in the a_3 direction, so we have *plane strain*.
The rules that determine u_1 and u_2 are that $u_1 \propto a_2$ and $u_2 \propto a_1$ with *different* constants of proportionality k; so:

$$u_1 = k_1 a_2 \text{ and } u_2 = k_2 a_1$$

Find the Strain and Rotation tensors, the equivalent rotation vector, the Principal Strains, the Principal Axes and the dilatation.

NB example with $k_1 = k_2$ is in lectures.

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Assignment 4 ***due 19 April.***

(1) Assuming the form of Hooke's Law for an isotropic material:

$$S_{ij} = 2\mu E_{ij} + \lambda E_{kk}\delta_{ij}$$

(i) Show that the Bulk Modulus K defined to be $1/3\, S_{kk} / E_{kk}$ is given by $K = \lambda + 2/3\,\mu$

Young's modulus Y is measured as the ratio of a uniaxial tension S_{11} to the strain E_{11} it produces in a body ('uniaxial' means that S_{22} and $S_{33} = 0$).

(ii) Write down the equations for S_{11}, S_{22} and S_{33} from Hooke's Law.

(iii) Under uniaxial tension, the body will contract in the x_2 and x_3 directions i.e. $E_{22}, E_{33} \neq 0$.

Poisson's Ratio ν is defined by: $\nu = - E_{22} / E_{11}$

Assuming that the body has axial symmetry, show that

$$\nu = \lambda / 2(\lambda + \mu)$$

(iv) Show that: $$Y = \mu (3\lambda + 2\mu)/(\lambda + \mu)$$

NB by making appropriate measurements of K, Y and ν, we can infer the Lame Constants for a material.

(2) If E_{ij} and S_{ij} are the strain and stress tensors in a continuum, the *strain potential energy* W per unit volume of the material is defined to be the work done in straining a unit volume of material to strain E_{ij}:

$$W = \int_0^{E_{ij}} S_{kl}\, dE_{kl} \quad \text{(summation convention).}$$

(i) Show that for an elastic material

$$W = \lambda/2\, E_{kk}^{\ 2} + \mu E_{kl} E_{kl}$$

3. (i) Show that $u_1 = A \sin(\omega t \pm \nu x_1)$; $u_2 = 0$; $u_3 = 0$

where t is time and A, ω and ν are constants, is a solution of Navier's equation without body forces viz.

$$\rho\, \partial^2 u_i / \partial t^2 = \mu\, \partial^2 u_i / \partial x_j \partial x_j + (\mu + \lambda)\, \partial^2 u_k / \partial x_k \partial x_i$$

provided $c = \omega/\nu$ satisfies

$$c^2 = (K + 4/3\,\mu)/\rho \qquad \text{where K is the Bulk Modulus.}$$

(ii) *Briefly* describe the way in which the continuum is deforming in this motion –

- In space (fix time)
- In time (fix position x_1).

(iii) If $(\omega t \pm \nu x_1)$ is dimensionless, what are the dimensions (units) for ω, ν and c? What is the physical meaning of c?

Tutorial Five 12 April (and 16 April if required)

1. A specimen of isotropic material is subjected to compression S_{11}, but is constrained so that it cannot expand in the x_2 and x_3 directions. Show that the apparent modulus of elasticity is:

$Y(1 - \nu)/(1 + \nu)(1 - 2\nu)$

Hence show that we can re-write Hooke's law for isotropic elastic solids as:

$$E_{ij} = (1 + \nu)/Y \; S_{ij} - \nu/Y \; S_{kk} \, \delta_{ij}$$

2. If E_{kl} is defined by $E_{12} = E_{21} = -2$, $E_{kl} = 0$ otherwise, evaluate

$$I = \int_0^{E_{kl}} U_{ij} \, dU_{ij}$$

3. Show that:

$u_2 = A \sin(\omega t \pm \nu x_1)$; $u_1 = 0$; $u_3 = 0$

is a solution of Navier's equation without body forces -

$$\rho \; \partial^2 u_i / \partial t^2 = \mu \, \partial^2 u_i / \partial x_j \partial x_j + (\mu + \lambda) \; \partial^2 u_k / \partial x_k \partial x_i$$

provided $c = \omega/\nu$ satisfies

$$c^2 = \mu/\rho$$