Swing High Module Assignment

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Due: 3.00pm, Fri 9 Aug 2013.

1. Prove that the pendulum equation

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0\ , \quad \theta(0) = a\ , \ \ \dot{\theta}(0) = 0$$

has periodic solutions with period

$$P = \frac{4}{\omega_0} \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}}$$

where

$$k^2 = \sin^2(a/2)\ .$$

Hint: multiply the differential equation by $\dot{\theta}$ and integrate once. Think about curves in the phase plane $(\theta, \dot{\theta})$. The work in Lin & Segel (LS) p.56 Ex.9 may also help.

2. Use the above exact period P, and expand for small a to verify the approximate improved period we obtained in lectures using Poincaré's method:

$$P \sim \frac{2\pi}{\omega_0} \left(1 + \frac{a^2}{16}\right)\ .$$

3. In biological applications the population P of certain organisms at time t is sometimes assumed to obey the logistic or Verhulst-Pearl equation

$$\frac{dP}{dt} = aP(1 - P/E) \tag{1}$$

where t is time, and E and a are positive constants.

   (a) Determine the equilibrium population levels.

   (b) Examine their stability.

   (c) Use this information to discuss the qualitative behaviour of the population (that is, what happens as time increases?). Reinforce your discussion by examining the levels of P at which P is increasing and decreasing, respectively.

   (d) Solve equation (1) exactly, and compare the results with your analysis in (a), (b) and (c) above.

4. Assume all constants are positive in this question:

(a) The equations

$$\frac{dx}{dt} = (a - bx - cy)x, \qquad \frac{dy}{dt} = (e - fx - gy)y,$$

are a simple model for the competition between two species of organisms. (Here a, b, c, e, f, and g are constants.) Write a brief essay on what is assumed in this model, and on what some of its limitations are expected to be.

(b) By examining the phase plane, show that if $(a/c) > (e/g)$ and $(a/b) > (e/f)$, then species x wins. Is this reasonable?

5. The equation for a damped harmonic oscillator is

$$m\ddot{x} + a\dot{x} + kx = 0 ,$$

where mass m, damping a, and spring constant k are all positive parameters. $x(t)$ is the position of the mass m attached to a spring.

(a) Write the equation as a system of coupled first-order ordinary differential equations by introducing $y = \dot{x}$.

(b) Show that $(x, y) = (0, 0)$ is a critical point.

(c) Describe the nature and stability of this critical point, and sketch solutions near origin in the phase plane, in the following cases:

(i) $a = 0$

(ii) $a^2 - 4km < 0$

(iii) $a^2 - 4km = 0$

(iv) $a^2 - 4km > 0$

(d) Interpret the results physically.