SCHOOL OF MATHEMATICS, STATISTICS, AND OPERATIONS RESEARCH
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MATH 466 — APPLIED MATHEMATICS — T1 and T2 2013

Module on Quantum Mechanics: Assignment 5

- This fifth assignment is specific to the honours-level quantum module (Math 466).
- You do *not* need to do this assignment if you are enrolled in 3rd-year Math 321/322/323.
- In assignment 3 you did some simple calculations describing transmission and reflection from a a compound barrier consisting of two identical sub-barriers separated by an adjustable distance; the present assignment will deal with unequal barriers and multiple barriers.
- Carefully read the article "Compound transfer matrices: Constructive and destructive interference" (Journal of Mathematical Physics, 2012), and answer the questions below.
- For extra background you could also take a look at the electronic preprint (e-print) "Bounds on variable-length compound jumps" (2013).
- Let me know of any typos.

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1. [Easy] As a warm-up, prove the following mathematical identities:

   (a) $\sinh\left(\sinh^{-1} A + \sinh^{-1} B\right) = A\,\sqrt{1+B^2} + \sqrt{1+A^2}\;B$.

   (b) $\cosh\left(\sinh^{-1} A + \sinh^{-1} B\right) = \sqrt{1+A^2}\,\sqrt{1+B^2} + A\;B$.

   (c) $\cosh\left(\cosh^{-1} A + \cosh^{-1} B\right) = A\;B + \sqrt{A^2-1}\,\sqrt{B^2-1}$.

   (d) $\tanh\left(\tanh^{-1} A + \tanh^{-1} B\right) = \dfrac{A+B}{1+AB}$.

   (e) $\operatorname{sech}\left(\operatorname{sech}^{-1} A + \operatorname{sech}^{-1} B\right) = \dfrac{AB}{1+\sqrt{1-A^2}\sqrt{1-B^2}}$.

2. [Easy]

Consider two barriers described (as in the notes, and in the JMP article) by transfer matrices

$$M_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ \beta_1^* & \alpha_1^* \end{bmatrix}; \qquad |\alpha_1|^2 - |\beta_1|^2 = 1,$$

and

$$M_2 = \begin{bmatrix} \alpha_2 & \beta_2 \\ \beta_2^* & \alpha_2^* \end{bmatrix}; \qquad |\alpha_2|^2 - |\beta_2|^2 = 1.$$

The compound two-barrier system will also be described by *some* transfer matrix

$$M_{12} = \begin{bmatrix} \alpha_{12} & \beta_{12} \\ \beta_{12}^* & \alpha_{12}^* \end{bmatrix}; \qquad |\alpha_{12}|^2 - |\beta_{12}|^2 = 1.$$

Assuming that the two sub-barriers are non overlapping:

(a) How would you calculate M_{12} in terms of M_1 and M_2?

(b) Explicitly calculate α_{12} in terms of α_1, β_1, α_2 and β_2.

(c) Explicitly calculate β_{12} in terms of α_1, β_1, α_2 and β_2.

3. [Easy]

From the explicit formula for α_{12} you have derived above, show how to deduce

$$|\alpha_1||\alpha_2| - |\beta_1||\beta_2| \leq |\alpha_{12}| \leq |\alpha_1||\alpha_2| + |\beta_1||\beta_2|.$$

4. [Straightforward]

From the explicit formula for β_{12} you have derived above, show how to deduce

$$\Big| |\alpha_1||\beta_2| - |\beta_1||\alpha_2| \Big| \leq |\beta_{12}| \leq |\alpha_1||\beta_2| + |\beta_1||\alpha_2|$$

5. [Straightforward]

Using the relationship between the Bogoliubov coefficient α and the transmission probability T, together with the normalization constraint $|\alpha|^2 - |\beta|^2 = 1$, show how to turn the bound on $|\alpha_{12}|$ into a bound on the transmission probability T_{12}.

Specifically, demonstrate that:

$$\mathrm{sech}^2\left\{\mathrm{sech}^{-1}\sqrt{T_1}+\mathrm{sech}^{-1}\sqrt{T_2}\right\}\leq T_{12}\leq \mathrm{sech}^2\left\{\mathrm{sech}^{-1}\sqrt{T_1}-\mathrm{sech}^{-1}\sqrt{T_2}\right\}.$$

6. [Straightforward]

   Using the relationship between the Bogoliubov coefficients (α and β) and the transmission probability T, together with the normalization constraint $|\alpha|^2-|\beta|^2=1$, show how to turn the bounds on $|\alpha_{12}|$ and $|\beta_{12}|$ into a bound on the refection probability R_{12}.

   Specifically, demonstrate that:

$$\tanh^2\left\{\tanh^{-1}\sqrt{R_1}-\tanh^{-1}\sqrt{R_2}\right\}\leq R_{12}\leq \tanh^2\left\{\tanh^{-1}\sqrt{R_1}+\tanh^{-1}\sqrt{R_2}\right\}.$$

7. [Easy]

   Using the hyperbolic trig identities proved in question 1, convert the bound on the transmission probability T to the form:

$$\frac{T_1T_2}{\left\{1+\sqrt{1-T_1}\sqrt{1-T_2}\right\}^2}\leq T_{12}\leq \frac{T_1T_2}{\left\{1-\sqrt{1-T_1}\sqrt{1-T_2}\right\}^2}.$$

8. [Easy]

   Using the hyperbolic trig identities proved in question 1, convert the bound on the reflection probability R to the form:

$$\left\{\frac{\sqrt{R_1}-\sqrt{R_2}}{1-\sqrt{R_1}\sqrt{R_2}}\right\}^2\leq R_{12}\leq \left\{\frac{\sqrt{R_1}+\sqrt{R_2}}{1+\sqrt{R_1}\sqrt{R_2}}\right\}^2.$$

9. [Straightforward]

   If we reinterpret the same mathematics in terms of a time-dependent parametrically excited oscillator, the same sort of logic can be used to obtain a bound on the number of particles cerated by parametric amplification.

   Use the relation between the Bogoliubov coefficient β and the number of particles N created by parametric amplification to deduce:

$$\sinh^2\left\{\sinh^{-1}\sqrt{N_1}-\sinh^{-1}\sqrt{N_2}\right\}\leq N_{12}\leq \sinh^2\left\{\sinh^{-1}\sqrt{N_1}+\sinh^{-1}\sqrt{N_2}\right\}.$$

10. [Easy]

    Using the hyperbolic trig identities proved in question 1, convert the bound on the number of created particles N to the form:

$$\left\{\sqrt{N_1(N_2+1)} - \sqrt{N_2(N_1+1)}\right\}^2 \leq N_{12} \leq \left\{\sqrt{N_1(N_2+1)} + \sqrt{N_2(N_1+1)}\right\}^2.$$

11. [Straightforward]

    Now consider n non-overlapping barriers in a row.

    Prove the straightforward result that:

$$|\alpha_{12\dots n}| \leq \cosh\left\{\sum_{i=1}^{n} \cosh^{-1}|\alpha_i|\right\}; \qquad |\beta_{12\dots n}| \leq \sinh\left\{\sum_{i=1}^{n} \sinh^{-1}|\beta_i|\right\}.$$

12. [Straightforward]

    Convert these bounds on $|\alpha_{12\dots n}|$ and $|\beta_{12\dots n}|$ to bounds on the transmission probability, the reflection probability, and (in the parametric oscillator interpretation) the number of created particles.

    Specifically, show that:

$$T_{12\dots n} \geq \operatorname{sech}^2\left\{\sum_{i=1}^{n} \operatorname{sech}^{-1}\sqrt{T_i}\right\};$$

$$R_{12\dots n} \leq \tanh^2\left\{\sum_{i=1}^{n} \tanh^{-1}\sqrt{R_i}\right\}.$$

$$N_{12\dots n} \leq \sinh^2\left\{\sum_{i=1}^{n} \sinh^{-1}\sqrt{N_i}\right\}.$$

13. [Difficult]

    Again consider n non-overlapping barriers in a row.

    Define the quantities

$$\Theta_{\text{peak}} = \max_{i\in\{1,2,3,\dots,n\}} \cosh^{-1}|\alpha_i|; \qquad \Theta_{\text{total}} = \sum_{i=1}^{n} \cosh^{-1}|\alpha_i|.$$

Now prove the decidedly non-trivial results that

$$|\alpha_{12\dots n}| \geq \cosh\left[\max\{2\Theta_{\text{peak}} - \Theta_{\text{total}}, 0\}\right];$$

$$|\beta_{12\dots n}| \geq \sinh\left[\max\{2\Theta_{\text{peak}} - \Theta_{\text{total}}, 0\}\right].$$

Doing this will require you to both *read* and *understand* most of the technical details of the article "Compound transfer matrices: Constructive and destructive interference" (Journal of Mathematical Physics, 2012).

Note minor changes in notation — this is deliberate, it is part of the assignment to force you to read and *comprehend* a research-level article.

14. [Trivial]

Using the definition of Θ_{total} above, and the results of question 12, justify the definitions

$$T_{12\dots n} \geq T_{\min} \equiv \operatorname{sech}^2\{\Theta_{\text{total}}\},$$

$$R_{12\dots n} \leq R_{\max} \equiv \tanh^2\{\Theta_{\text{total}}\}.$$

$$N_{12\dots n} \leq N_{\max} \equiv \sinh^2\{\Theta_{\text{total}}\},$$

15. [Straightforward]

Using the definitions and bounds of the previous two questions, show that

$$T_{12\dots n} \leq \operatorname{sech}^2\left[\max\left\{2\operatorname{sech}^{-1}\sqrt{T_{\text{peak}}} - \operatorname{sech}^{-1}\sqrt{T_{\min}}, 0\right\}\right].$$

$$R_{12\dots n} \geq \tanh^2\left[\max\left\{2\tanh^{-1}\sqrt{R_{\text{peak}}} - \tanh^{-1}\sqrt{R_{\max}}, 0\right\}\right].$$

$$N_{12\dots n} \geq \sinh^2\left[\max\left\{2\sinh^{-1}\sqrt{N_{\text{peak}}} - \sinh^{-1}\sqrt{N_{\max}}, 0\right\}\right].$$

- That's all — please let me know of any typos or obscurities.
- For extra background you could also take a look at the electronic preprint (e-print) "Bounds on variable-length compound jumps" (2013).

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