School Of Mathematics, Statistics, and Operations Research
Te Kura Mātai Tatauranga, Rangahau Pūnaha

MATH 321/322/323 Applied Mathematics T1 and T2 2013

Module on Quantum Mechanics: Assignment 2

- This second assignment will deal with classical tunnelling, and its relation to quantum tunnelling.
- Read chapter 4 of the notes — the chapter on tunnelling.
- The assignment will lead you through the details of "frustrated total internal reflection" in the case of fluid acoustics.
- The individual steps are very easy.
- However, *you will actually need to think...*
- Problem 5 is a little tedious — *but it is utterly straightforward...*
- Please let me know if you find typos in the notes or assignment.

---

1. Consider a sound wave in a fluid medium (for example, air, water).

   Adopt the formalism of section 4.4.1 (total internal reflection in fluid-fluid acoustics).

   The mean-square pressure fluctuation (defined by taking a time-average over one period T of the oscillating pressure field) is *defined* by

   $$(\Delta p)^2 = \frac{1}{T}\int_0^T [\mathbf{Re}(p)]^2 \, \mathrm{d}t.$$

   Show that in medium 1, where the incoming wave is defined to be

   $$p = p_1 \exp\left[-i\{\omega t - \vec{k}_1 \cdot \vec{x}\}\right],$$

the mean-square pressure fluctuation of this incoming wave is

$$(\Delta p)^2 = \frac{1}{2}\, |p_1|^2.$$

(This is standard textbook work. Remember the dim dark ages when you worked with "RMS" [root mean square] voltages and currents? The basic ideas are the same.)

2. Now, using the results presented in the notes, show that in medium 2 the mean square pressure fluctuation is

$$(\Delta p)^2 = \frac{1}{2}\, |p_2|^2 \, \exp\left[2\, ||\vec{k}_1||\; \mathbf{Re}\left(\sqrt{\sin^2\theta_1 - \sin^2\theta_*} \right)\; z \right],$$

where p_2 is still to be determined and this expression has been carefully written in such a way that it remains valid in all cases — either with or without total internal reflection.

3. Now introduce a *three-layer* system, with three distinct speeds of sound, c_1, c_2, and c_3.

   Assume two planar interfaces, located at $z = 0$ and $z = L$ respectively.

   Adopt the notation of section 4.4.2 (frustrated total internal reflection in fluid-fluid acoustics).

   Fill out the details in the derivation of the four boundary conditions:

$$p_1 + p_1^R = p_2 + p_2^R;$$

$$\kappa_1\{p_1 - p_1^R\} = \kappa_2\{p_2 - p_2^R\};$$

$$p_2\; \exp[i\kappa_2 L] + p_2^R\; \exp[-i\kappa_2 L] = p_3\; \exp[i\kappa_3 L];$$

$$\kappa_2 \left\{ \; p_2\; \exp[i\kappa_2 L] - p_2^R\; \exp[-i\kappa_2 L] \right\} = \kappa_3\; p_3\; \exp[i\kappa_3 L].$$

4. Now:

   (a) Show that κ_1 is by construction always guaranteed to be pure real.

   (b) Show that κ_2 and κ_3 are by construction always guaranteed to be either pure real or pure imaginary.

   (At least as long as the refractive index is real, and let's not open that particular rat's nest...)

5. Solve the above set of linear equations and explicitly evaluate

$$p_3 = p_1\ F(\kappa_1, \kappa_2, \kappa_3, L).$$

You can use Maple, Mathematica, or any other computer-aided symbolic manipulation system, or you could do this *by hand*.

(This will be tedious, though it is not intrinsically difficult.)

(This is a *really good excuse* to learn some Maple; you can use the computers in the Mac Lab on the 4th floor.)

(Note that Maple can quite literally be "thick as a brick". I do expect you to use a little common sense and human insight in massaging and presenting whatever result you extract from Maple.)

6. Explicitly evaluate the transmission coefficient (the ratio of transmitted power to incident power)

$$T = \frac{P_3}{P_1}.$$

First show that

$$T = \frac{P_3}{P_1} = |F(\kappa_1, \kappa_2, \kappa_3, L)|^2 \ \ \mathbf{Re}\left(\frac{\kappa_3}{\kappa_1}\right),$$

and then explicitly evaluate this in terms of κ_1, κ_2, κ_3, and L.

You can use Maple, Mathematica, or any other computer-aided symbolic manipulation system, or you could do this *by hand*.

(If you do use Maple, I do expect you to use a little common sense and human insight in massaging and presenting whatever result you extract from Maple.)

7. Consider the special case $n_3 = n_1$, (so $c_3 = c_1$), where we also have $\kappa_3 = \kappa_1$.

In this special case first show that

$$T = \frac{P_3}{P_1} = |F(\kappa_1, \kappa_2, \kappa_1, L)|^2,$$

and then explicitly evaluate the transmission coefficient in terms of κ_1, κ_2, and L.

8. Now re-write this special case transmission coefficient in terms of c, ω, $\sin\theta_1$, $\sin\theta_*$, and L.

9. **Extra credit:** Compare the transmission coefficients calculated above with those that you get from a quantum mechanical particle scattering off a two-step potential.

   (You will need to look up your elementary quantum textbooks for this, and/or find a few slightly more detailed textbooks on quantum physics, and/or do a little digging on Wikipedia or Google.

   It really should not be all that hard.)

— # # # —