School Of Mathematics, Statistics, and Operations Research
Te Kura Mātai Tatauranga, Rangahau Pūnaha

MATH 466 | Applied Mathematics | T1 and T2 2013

Module on Mechanix: Assignment 5

- This fifth assignment is specific to the honours-level mechanix module (Math 466).
- You do *not* need to do this assignment if you are enrolled in 3rd-year Math 321/322/323.
- Let me know of any typos.

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1. **Elliptical orbits in Newtonian gravity:**

   In previous assignments we have already seen that

   $$\text{(circular orbits)} + \text{(Kepler's laws)} \Rightarrow \text{(inverse square law)}.$$

   We now want to check that

   $$\text{(inverse square law)} \Rightarrow \text{(elliptic orbits)},$$

   or more generally

   $$\text{(inverse square law)} \Rightarrow \text{(circular/elliptic/parabolic/hyperbolic orbits)},$$

   or even more generally

   $$\text{(Kepler's laws)} \Leftrightarrow \text{(inverse square law)}.$$

   Note that this result goes all the way back to Newton — and in fact establishing this particular, result (inverse square law) ⇒ (elliptic orbits), was the primary reason Newton developed his version of the differential and integral calculus.

We have already seen how to decompose the gravitational 2-body problem into a trivial centre of mass (COM) motion plus a nontrivial relative motion.

Discard the trivial COM motion, and focus on the relative motion.

In terms of the reduced mass μ and total mass M the Lagrangian for the relative motion in Newtonian gravity has already been shown to simplify to

$$L = \frac{1}{2}\mu \left|\dot{\vec{x}}\right|^2 + \frac{G\mu M}{|\vec{x}|}.$$

(a) [Trivial] Using the Euler–Lagrange equations, verify that the resulting equation of motion is the standard inverse-square law

$$\ddot{\vec{x}} = -\frac{GM}{|\vec{x}|^2}\,\hat{x}.$$

(b) [Trivial] Verify that the energy

$$E = \frac{1}{2}\mu \left|\dot{\vec{x}}\right|^2 - \frac{G\mu M}{|\vec{x}|}$$

is conserved.

(Note the minus sign; it is important.)

(c) [Trivial] Verify that the angular momentum

$$\vec{J} = \vec{p} \times \vec{x} = \mu\, \dot{\vec{x}} \times \vec{x}$$

is conserved.

(d) [Trivial] Show that, since the angular momentum $\vec{J}$ is conserved, one can without loss of generality choose coordinates to make $\vec{J}$ point along the z axis.

(e) [Trivial] If this is done, argue that the position and velocity can always be chosen to lie purely in the (x, y) plane:

$$\vec{x} = (x, y, 0); \qquad \dot{\vec{x}} = (\dot{x}, \dot{y}, 0)\,.$$

(f) [Easy] Adopt polar coordinates (r, θ) so that we have

$$\vec{x} = (x, y, 0) = (r\cos\theta, r\sin\theta, 0) = r(\cos\theta, \sin\theta, 0).$$

Show that

$$\dot{\vec{x}} = (\dot{x}, \dot{y}, 0) = \dot{r}(\cos\theta, \sin\theta, 0) + r(-\sin\theta, \cos\theta, 0)\dot{\theta},$$

and that

$$\ddot{\vec{x}} = (\ddot{x}, \ddot{y}, 0) = [\ddot{r} - r\dot{\theta}^2](\cos\theta, \sin\theta, 0) + [2\dot{r}\dot{\theta} + r\ddot{\theta}](-\sin\theta, \cos\theta, 0).$$

(g) [Easy] Hence verify that the equations of motion reduce to

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}; \qquad 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0.$$

(h) [Easy] Verify that the second of these equations is equivalent to the constancy of

$$|\vec{J}| = \mu r^2 \dot{\theta}.$$

(i) [Easy] Hence show that

$$\ddot{r} = \frac{J^2}{\mu^2 r^3} - \frac{GM}{r^2}.$$

You could in principle integrate this ODE directly — good luck.

(j) [Devious] Instead let's be a little devious — write $r = r(\theta)$ and show that

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\,\frac{J}{\mu r^2}.$$

(k) [Devious] Thence show

$$\begin{aligned}
\ddot{r} &= \frac{d}{d\theta}\left(\frac{dr}{d\theta}\frac{J}{\mu r^2}\right)\dot{\theta} \\
&= \frac{d}{d\theta}\left(\frac{dr}{d\theta}\frac{J}{\mu r^2}\right)\frac{J}{mr^2} \\
&= \frac{d^2 r}{d\theta^2}\frac{J^2}{\mu^2 r^4} - 2\left(\frac{dr}{d\theta}\right)^2\frac{J^2}{\mu^2 r^5}.
\end{aligned}$$

(l) [Devious] Now let's be even more devious — write

$$u(\theta) = \frac{1}{r(\theta)},$$

and show that
$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta},$$
and that
$$\frac{d^2u}{d\theta^2} = -\frac{1}{r^2}\frac{d^2r}{d\theta^2} + \frac{2}{r^3}\left(\frac{dr}{d\theta}\right)^2.$$

(m) [Easy] Hence deduce
$$\ddot{r} = -\frac{J^2}{m^2r^2}\frac{d^2u}{d\theta^2} = -\frac{J^2u^2}{m^2}\frac{d^2u}{d\theta^2}.$$

(n) [Easy] Inserting this into the radial equation of motion deduce
$$-\frac{J^2u^2}{\mu^2}\frac{d^2u}{d\theta^2} = \frac{J^2u^3}{\mu^2} - GMu^2,$$
and from this obtain
$$\frac{d^2u}{d\theta^2} = \frac{GM\mu^2}{J^2} - u.$$

(o) [Easy] The virtue of taking these extremely devious intermediate steps is that this last ODE is now very easy to integrate.
Define
$$\tilde{u} = u - \frac{GM\mu^2}{J^2},$$
so that
$$\frac{d^2\tilde{u}}{d\theta^2} = -\tilde{u}.$$
Show that the general solution to this last ODE is:
$$\tilde{u}(\theta) = A\cos(\theta + B).$$

(p) [Easy] From this deduce
$$u(\theta) = \frac{GM\mu^2}{J^2} + A\cos(\theta + B),$$
whence
$$r(\theta) = \frac{1}{\frac{GM\mu^2}{J^2} + A\cos(\theta + B)},$$

which one can rewrite as

$$r(\theta) = \frac{J^2}{GM\mu^2} \frac{1}{1 + e\cos(\theta + B)},$$

or even better as

$$r(\theta) = \frac{J^2}{GM\mu^2(1-e^2)} \frac{1-e^2}{1 + e\cos(\theta + B)}.$$

Recognize that this is one of the standard forms of representing an ellipse (with polar coordinates relative to one of the foci of the ellipse).

Remember Kepler's first law: the planets move in ellipses with the sun at one focus.

(A more precise statement is that the planets move in ellipses with the 2-body center of mass at one focus).

The quantity e is the eccentricity of the ellipse.

The quantity

$$a = \frac{J^2}{GM\mu^2}$$

is called the **semi latus rectum** of the ellipse.

(q) [Easy] Calculate the semi major axis of the ellipse.

(r) [Easy] Calculate the semi minor axis of the ellipse.

(s) [Easy] What happens if $e = 0$?
Physically interpret this situation.

(t) [Easy] What happens if $e = 1$?
Physically interpret this situation.

(u) [Easy] What happens if $e > 1$?
Physically interpret this situation.
(Yes, this does happen in the "real world".)

I realise this has been somewhat painful — but just think what Newton had to do when coming up with an equivalent argument *and* inventing calculus at the same time.

2. **Virial theorem:**

The so-called virial theorem is most often formulated and used within the context of non-relativistic mechanics of a n-body system interacting via central forces.

Let us consider the Lagrangian

$$L = T - V_{\text{total}},$$

where

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \left|\dot{\vec{x}}_i\right|^2; \qquad \text{and} \qquad V_{\text{total}} = \sum_{i<j} V\left(|\vec{x}_i - \vec{x}_j|\right).$$

Define quantities called the "scalar moment of inertia" I, and the "scalar virial" G, by:

$$I = \sum_{i=1}^{n} m_i |\vec{x}_i|^2; \qquad \text{and} \qquad G = \sum_{i=1}^{n} \vec{p}_i \cdot \vec{x}_i = \sum_{i=1}^{n} m_i\, \dot{\vec{x}}_i \cdot \vec{x}_i.$$

(a) [Easy]
Assuming the individual masses are constant show that

$$\frac{dI}{dt} = 2G.$$

(b) [Easy]
Show

$$\frac{dG}{dt} = 2T + \sum_{i=1}^{n} \vec{F}_i \cdot \vec{x}_i.$$

(c) [Straightforward]
Define $r_{ij} = |\vec{x}_i - \vec{x}_j|$.
Using the fact that V_{total} is assumed to be sum of 2-body central potentials, demonstrate that

$$\frac{dG}{dt} = 2T - \sum_{i<j}^{n} \left.\frac{dV}{dr}\right|_{r_{ij}} r_{ij}.$$

(d) [Easy]

For a power law potential $V(r) = \alpha r^\beta$ show that this implies

$$\frac{dG}{dt} = 2T - \beta V_{\text{total}}.$$

(e) [Easy]

In particular, for n particles interacting via Newtonian gravity or electrostatic forces show that this implies

$$\frac{dG}{dt} = 2T + V_{\text{total}}.$$

(f) [Easy]

If the system is assumed to undergo periodic motion show that the time average of dG/dt vanishes identically:

$$\left\langle \frac{dG}{dt} \right\rangle = 0.$$

Note: Even if the motion is not exactly periodic there are still situations under which one can usefully *approximate*

$$\left\langle \frac{dG}{dt} \right\rangle \approx 0.$$

(g) [Easy]

Under the assumption of periodic motion under a power-law potential $V(r) = \alpha r^\beta$ show:

$$\langle T \rangle = \frac{\beta}{2} \langle V_{\text{total}} \rangle.$$

(h) [Easy]

Under the assumption of periodic motion under Newtonian gravity or electrostatic forces show:

$$\langle T \rangle = -\frac{1}{2} \langle V_{\text{total}} \rangle.$$

(i) [Tricky]

What if anything can you say about the situation where the particles are relativistic?

Consider the quantity

$$T_i = \frac{m_i c^2}{\sqrt{1 - v^2/c^2}} - m_i c^2,$$

and find an appropriate virial theorem.

Can you generalize this even further?

(j) [Tricky]

What (if anything) can you say about the situation where the 2-body forces are not a power law?

End of honours-level assignment for the mechanix module.

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