School Of Mathematics, Statistics, and Operations Research

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MATH 321/322/323 — Applied Mathematics — T1 and T2 2013

Module on Mechanix: Assignment 4

- This fourth and final assignment will deal with the interplay between the Hamiltonian and Lagrangian descriptions of classical mechanics.
- By now you should have read all the notes.
- Applied math level proofs are good enough — I do not demand absolute analytical rigour, but do want to see enough information so that any interested reader could fill in any missing steps.
- If you find any typos in the notes or assignment, please let me know.

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1. Consider a Hamiltonian of the form

$$H(\vec{p}, \vec{x}) = \sqrt{m_0^2\, c^4 + |\vec{p}|^2\, c^2} + V(\vec{x}).$$

(You should find this question "straightforward", it is closely related to the last question in Assignment 3; I have broken things down into simple little steps.)

(a) You will already have written down Hamilton's equations for this particular Hamiltonian, and you will have solved for $\vec{p}$ as a function of $\dot{\vec{x}}$, (and c and m_0).

Rewrite those key results.

(b) Now perform the transformation discussed in the notes whereby you can take a Hamiltonian and transform it into an equivalent Lagrangian.

Explicitly evaluate

$$L(\vec{x}, \dot{\vec{x}})$$

as a function of m_0, c, $\dot{\vec{x}}$, and $V(\vec{x})$.

(c) Find the Euler–Lagrange equations coming from this Lagrangian $L(\vec{x}, \dot{\vec{x}})$. Interpret them physically.

(d) Take the limit $|\dot{\vec{x}}| \ll c$ in the Euler–Lagrange equations you have just derived. What do you get? Interpret your result.

(e) Take the limit $|\dot{\vec{x}}| \ll c$ directly in the Lagrangian you derived above. Be certain to keep the first non-trivial term in $|\dot{\vec{x}}|$.

What do you get? Interpret your result.

(f) Again take the limit $|\dot{\vec{x}}| \ll c$ directly in the Lagrangian you derived above, but now keep the first *two* nontrivial terms in $\dot{\vec{x}}$.

What do you get? Interpret your result.

2. Consider a Lagrangian of the form:

$$L = \frac{1}{2} \left\{ \sum_{i,j=1}^{N} m_{ij}\, \dot{x}^i\, \dot{x}^j \right\} - V(x^k).$$

(a) Find the momenta p_i as functions of m_{ij} and $\dot{x}^i$.

(b) Rewrite this relation between the p_i and the $\dot{x}^i$ as a linear matrix equation involving the matrix $M = [m_{ij}]$.

(c) Invert this relationship to find the velocities $\dot{x}^i$ as functions of the momenta p_i and the quantities m_{ij}.

(You may find it useful to adopt the notation that m^{ij} stands for the matrix inverse M^{-1} of the $N \times N$ matrix $M = [m_{ij}]$ — note carefully the use of superscripts rather than subscripts — this is one of those tricky notational conventions that looks a little odd the first time you see it, but is ultimately *extremely* useful, and is also *extremely* common and standard.)

(d) Using these results, find the equivalent Hamiltonian $H(p_i, x^i)$.

(e) Develop a good physical interpretation for the $N \times N$ matrix m_{ij}.

3. Consider the Hamiltonian

$$H(\vec{p}, \vec{x}) = \frac{|\vec{p} - q\, \vec{A}(\vec{x})|^2}{2m} + q\, V(\vec{x})$$

that we looked at in the previous assignment.

(a) Show that this is equivalent to the Lagrangian

$$L(\vec{p},\vec{x}) = \frac{1}{2}m\left(\dot{\vec{x}}\right)^2 - q\left\{V(\vec{x}) + \dot{\vec{x}}\cdot\vec{A}(\vec{x})\right\}.$$

(b) Write down the Euler–Lagrange equations.

(c) Physically interpret these results.

4. Consider the Hamiltonian

$$H(\vec{p},\vec{x}) = \sqrt{m_0^2c^4 + c^2|\vec{p}-q\vec{A}|^2} + q\,V(\vec{x}).$$

This is the relativistic generalization of the Hamiltonian in the previous question.

(a) Demonstrate that if $|\vec{p}-q\vec{A}|^2 \ll m_0^2c^2$ this relativistic Hamiltonian will reduce to the non-relativistic Hamiltonian of the previous question.

(b) Show that this is equivalent to the Lagrangian

$$L(\vec{p},\vec{x}) = -m_0\sqrt{1-\left(\dot{\vec{x}}\right)^2/c^2} + q\left\{V(\vec{x}) + \dot{\vec{x}}\cdot\vec{A}(\vec{x})\right\}.$$

(c) Demonstrate that if $(\dot{\vec{x}})^2 \ll c^2$ this relativistic Lagrangian will reduce to the non-relativistic Lagrangian of the previous question.

(d) Write down the Euler–Lagrange equations for the full relativistic Lagrangian.

(e) How do they differ from the Euler–Lagrange equations of the non-relativistic Lagrangian of the previous question.

(f) Physically interpret these results.

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- End of fourth and final assignment in the undergraduate version of the Mechanix module.
- If you are taking this module as part of Honours-level Applied Math, be sure to complete the additional Assignment 5.

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