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MATH 312/322/323	Applied Mathematics	T1 and T2 2013

Module on Mechanix: Assignment 1

- This first assignment will deal with several simple Lagrangian systems.
- Read the chapters on the notes that deal with the Lagrangian formulation.
- Applied math level proofs are good enough — I do not demand absolute analytical rigour, but do want to see enough information so that any interested reader could fill in any missing steps.
- Note that there will be a lot of three-dimensional vectors occurring in this assignment — I expect you to carefully keep track of the vector nature of appropriate quantities when it is important. (Blindly pretending that everything can be reduced to one-dimensional physics is not an option.)
- If you find any typos in the notes or assignment, please let me know.

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1. Consider a mechanical system that consists of two charged particles interacting via the electric Coulomb potential.

   We shall build up the Lagrangian in stages.

   (a) Let the two particles be at positions $\vec{x}_1(t)$ and $\vec{x}_2(t)$, and have masses m_1 and m_2 respectively.
   What is the (non-relativistic) kinetic energy?

   (b) Let the two particles have electric charges q_1 and q_2 respectively.

   i. According to Coulomb's law, what is the force $\vec{F}_{1\to 2}$ that particle 1 exerts on particle 2?

ii. According to Coulomb's law, what is the force $\vec{F}_{2\to 1}$ that particle 2 exerts on particle 1?

iii. What, therefore, is the electric potential energy?

(c) From these two pieces of information, write down the Lagrangian for this 2-body system.

Note that the Lagrangian will depend on both m_1 and m_2, the time derivatives $\dot{\vec{x}}_1(t)$, and $\dot{\vec{x}}_2(t)$, the positions $\vec{x}_1(t)$ and $\vec{x}_2(t)$, and the electric charges q_1 and q_2, plus the proportionality constant that appears in Coulomb's law.

(d) Check that when you evaluate the Euler–Lagrange equations for this Lagrangian you reproduce (for each of the two particles individually) the full 3-dimensional Newton's equations of motion for a force determined by Coulomb's law.

2. Using the Lagrangian you have defined above, separate out the centre-of-mass motion from the relative motion of the two particles.

That is: perform the change of variables

$$\vec{x}_{\text{com}} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1+m_2}; \qquad \vec{x}_{\text{diff}} = \vec{x}_2 - \vec{x}_1.$$

(a) If you do the change of variables correctly the Lagrangian will split into two separate pieces:

$$L(\dot{\vec{x}}_1, \vec{x}_1, \dot{\vec{x}}_2, \vec{x}_2) = L_{\text{com}}(\dot{\vec{x}}_{\text{com}}, \vec{x}_{\text{com}}) + L_{\text{diff}}(\dot{\vec{x}}_{\text{diff}}, \vec{x}_{\text{diff}}).$$

Explicitly evaluate the two individual pieces $L_{\text{com}}(\dot{\vec{x}}_{\text{com}}, \vec{x}_{\text{com}})$ and $L_{\text{diff}}(\dot{\vec{x}}_{\text{diff}}, \vec{x}_{\text{diff}})$.

(b) Using the Euler–Lagrange equations for $L_{\text{com}}(\dot{\vec{x}}_{\text{com}}, \vec{x}_{\text{com}})$ find the equations of motion for $\vec{x}_{\text{com}}$. Interpret your result.

(c) Using the Euler–Lagrange equations for $L_{\text{diff}}(\dot{\vec{x}}_{\text{diff}}, \vec{x}_{\text{diff}})$ find the equations of motion for $\vec{x}_{\text{diff}}$. Interpret your result.

(d) You will find it useful to introduce a quantity

$$\mu = \frac{1}{\frac{1}{m_1}+\frac{1}{m_2}} = \frac{m_1 m_2}{m_1+m_2}.$$

This quantity is traditionally called the "reduced mass".

Why is it important?

3. Consider a mechanical system that consists of massive particles interacting via Newtonian gravity.

   This question will be very simple once you have done question 1 above; but there are (only a few) simple modifications.

   We shall build up the Lagrangian in stages.

   (a) Let the two particles be at positions $\vec{x}_1(t)$ and $\vec{x}_2(t)$, and have masses m_1 and m_2 respectively.

      What is the (non-relativistic) kinetic energy?

   (b) Now:

      i. According to Newton's law of gravitational attraction, what is the force $\vec{F}_{1\to 2}$ that particle 1 exerts on particle 2?

      ii. According to Newton's law of gravitational attraction, what is the force $\vec{F}_{2\to 1}$ that particle 2 exerts on particle 1?

      iii. What, therefore, is the gravitational potential energy?

   (c) From these two pieces of information, write down the Lagrangian for this 2-body system.

      Note that the Lagrangian will depend on both m_1 and m_2, the time derivatives $\dot{\vec{x}}_1(t)$, and $\dot{\vec{x}}_2(t)$, the positions $\vec{x}_1(t)$ and $\vec{x}_2(t)$, plus the proportionality constant G that appears in Newton's law of gravitational attraction.

   (d) Check that when you evaluate the Euler–Lagrange equations for this Lagrangian you reproduce (for each of the two particles individually) the 3-dimensional Newton's equations of motion for a force determined by Newton's law of gravitational attraction.

4. Using the Lagrangian you have defined above, now relevant for two bodies interacting *gravitationally*, again separate out the centre-of-mass motion from the relative motion of the two particles.

   (This question is very similar to question 2, but there is one key difference that I specifically want you to look out for.)

   That is: perform the change of variables

$$\vec{x}_{\text{com}} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2}; \qquad \vec{x}_{\text{diff}} = \vec{x}_2 - \vec{x}_1.$$

- If you do the change of variables correctly the Lagrangian will split into two separate pieces:

$$L(\dot{\vec{x}}_1, \vec{x}_1, \dot{\vec{x}}_2, \vec{x}_2) = L_{\text{com}}(\dot{\vec{x}}_{\text{com}}, \vec{x}_{\text{com}}) + L_{\text{diff}}(\dot{\vec{x}}_{\text{diff}}, \vec{x}_{\text{diff}}).$$

  Explicitly evaluate the two individual pieces $L_{\text{com}}(\dot{\vec{x}}_{\text{com}}, \vec{x}_{\text{com}})$ and $L_{\text{diff}}(\dot{\vec{x}}_{\text{diff}}, \vec{x}_{\text{diff}})$.
- Using the Euler–Lagrange equations for $L_{\text{com}}(\dot{\vec{x}}_{\text{com}}, \vec{x}_{\text{com}})$ find the equations of motion for $\vec{x}_{\text{com}}$. Interpret your result.
- Using the Euler–Lagrange equations for $L_{\text{diff}}(\dot{\vec{x}}_{\text{diff}}, \vec{x}_{\text{diff}})$ find the equations of motion for $\vec{x}_{\text{diff}}$. Interpret your result.
- For this particular problem, the "reduced mass"

$$\mu = \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}} = \frac{m_1 m_2}{m_1 + m_2}$$

  is *unlikely* to be either interesting or useful.

  What (related but distinct) quantity now plays a similar role in controlling the relative motion?

5. Write down the Lagrangian for *three* bodies interacting via Newton's law of gravitational attraction.

   (There is no way you will be able to *solve* this "three-body problem", even though you should very easily be able to write down a very simple looking Lagrangian for this system.)

   Explain the basic physics behind all the terms in the Lagrangian you have written down.

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