PDEs for earth Science: Diffusion equation part 2

Variable surface temperature

We will conclude the topic on the Diffusion equation with examination of a famous problem in geophysics: what is the temperature distribution in the Earth due to diurnal (daily) and seasonal variations in surface heating?

Again we will solve a general problem and then use it to tackle the specific one.

The problem is to determine $T(x, t)$ in the half-space $x \geq 0$ given a variable, *periodic* surface temperature $T(0,t) = \phi(t)$ when $T(x, t)$ satisfies

$$\partial T / \partial t - K \nabla^2 T = 0$$

In this problem we ignore the heat flux from the deep Earth (which, on a timescale of years, is constant, implying a small linear temperature gradient at the surface).

We also require that $T(x, t)$ is bounded in the half-space for all times.

Digression and revision: Fourier series

$\phi(t)$ is a real function with period $\tau = 2\pi/\omega$.

That is, $\phi(t) = \phi(t + n\tau)$ for all integers n (+, - and 0).

Then

$$\phi(t) = \sum_{n=0}^{\infty} A_n \cos(n \omega t) + \sum_{n=0}^{\infty} B_n \sin(n \omega t)$$

where A_n and B_n are given by:

$$A_0 = 1/\tau \int_{-\tau/2}^{\tau/2} \phi(t)\, dt$$

$$A_n = 2/\tau \int_{-\tau/2}^{\tau/2} \phi(t) \cos(n \omega t)\, dt \quad n \neq 0$$

$$B_n = 2/\tau \int_{-\tau/2}^{\tau/2} \phi(t) \sin(n \omega t)\, dt$$

for all integers n.

Lemma: Orthogonality relations

(i) $$1/\tau \int_{-\tau/2}^{\tau/2} \cos(n \omega t) \cos(m \omega t)\, dt$$

$= 0$ if $m \neq n$, $= \frac{1}{2}$ if $m = n \neq 0$, $= 1$ if $m = n = 0$

(ii) $$1/\tau \int_{-\tau/2}^{\tau/2} \sin(n\,\omega\, t)\sin(m\,\omega\, t)\,dt$$

$= 0$ if $m \neq n$, $= \frac{1}{2}$ if $m = n \neq 0$, $= 0$ if $m = n = 0$

(iii) $$1/\tau \int_{-\tau/2}^{\tau/2} \sin(n\,\omega\, t)\cos(m\,\omega\, t)\,dt$$

$= 0$ all m, n

The proof is straightforward but tedious. We need the trigonometric identities:

$$\cos A \cos B = \tfrac{1}{2}\,[\cos(A - B) + \cos(A + B)]$$

$$\sin A \sin B = \tfrac{1}{2}\,[\cos(A - B) - \cos(A + B)]$$

$$\sin A \cos B = \tfrac{1}{2}\,[\sin(A - B) + \sin(A + B)]$$

Then eg

$$\cos(n\,\omega\, t)\cos(m\,\omega\, t) = \tfrac{1}{2}\cos([n - m]\,\omega\, t) + \cos([n + m]\,\omega\, t)]$$

and for $m \neq n$,

$$1/\tau \int_{-\tau/2}^{\tau/2} \cos(n\,\omega\, t)\cos(m\,\omega\, t)\,dt$$

$$= 1/2\tau \int_{-\tau/2}^{\tau/2} \{\cos([n - m]\,\omega\, t) + \cos([n + m]\,\omega\, t)\}\,dt$$

$$= 1/2\tau \left[(1/[n - m]\,\omega)\sin([n - m]\,\omega\, t) + (1/[n + m]\,\omega)\sin([n + m]\,\omega\, t) \right]_{-\tau/2}^{\tau/2}$$

But $\omega\tau = 2\pi$ and so the sin terms are both zero, and therefore so is the integral.

If $m = n \neq 0$, we use $\cos^2 n\,\omega\, t = \frac{1}{2}\,[1 + \cos(2\,n\,\omega\, t)]$ to get the required result.

For $m = n = 0$ the result is immediate.

Fourier coefficients for $\phi(t)$

What choice of A_n, B_n will make

$$\phi(t) - \left\{ \sum_{n=0}^{\infty} A_n \cos(n\,\omega\, t) + \sum_{n=0}^{\infty} B_n \sin(n\,\omega\, t) \right\} = 0\ ?$$

Integrate the LHS from $t = -\tau/2$ to $t = \tau/2$:

$$1/\tau \int_{-\tau/2}^{\tau/2} \phi(t) \cos(n \omega t)\, dt - 1/\tau \int_{-\tau/2}^{\tau/2} \sum_{k=0}^{\infty} A_k \cos(k \omega t) \cos(n \omega t)\, dt$$

$$- 1/\tau \int_{-\tau/2}^{\tau/2} \sum_{k=0}^{\infty} B_k \sin(k \omega t) \cos(n \omega t)\, dt$$

By the Lemma the second term = 0 for all n, k. The first term = 0 if $n \neq k$, so the integral is

$$1/\tau \int_{-\tau/2}^{\tau/2} \phi(t) \cos(n \omega t)\, dt - A_n/2 \quad \text{for } n \neq 0$$

or

$$1/\tau \int_{-\tau/2}^{\tau/2} \phi(t) \cos(n \omega t)\, dt - A_0 \quad \text{for } n = 0$$

So

$$A_n = 2/\tau \int_{-\tau/2}^{\tau/2} \phi(t) \cos(n \omega t)\, dt$$

$$A_0 = 1/\tau \int_{-\tau/2}^{\tau/2} \phi(t)\, dt$$

and similarly

$$B_n = 2/\tau \int_{-\tau/2}^{\tau/2} \phi(t) \sin(n \omega t)\, dt$$

are *necessary* conditions for the integral to be zero and hence for

$$\phi(t) = \sum_{n=0}^{\infty} A_n \cos(n \omega t) + \sum_{n=0}^{\infty} B_n \sin(n \omega t) = 0 \qquad (4)$$

Sufficiency takes some more effort. In particular we need to worry about the conditions under which the infinite series converges. See any good text on PDEs eg “Partial Differential Equations and boundary value problems with applications” Mark A Pinsky, McGraw-Hill 3ed; QA374 P658 p.

We will skip this and presume we can represent any function $\phi(t)$ of interest by equation (4)

Solution

We can solve the problem using our separated solution:

$T(x, t) = X(x)\, C(t)$, where now there is no y or z dependence (NB flat Earth!).

$$(1/X)\, d^2X/ dx_1^2 = \nu^2 ,$$

$$(1/C)\, dC/dt = K \nu^2 = - \lambda.$$

So C, X are given by:

$$C(t) = C_0 \exp(- \lambda t).$$

$$X(x) = X_0\exp(\nu x) + X_1\exp(- \nu x)$$

Now T(x, t) being bounded *rules out λ being real*, as if it is either positive or negative $\exp(- \lambda t)$ will increase for one of $t \to \infty$ or $\to -\infty$ (going backward in time) .

So write $\lambda = i \omega$. Then

$$\nu^2 = - i \omega /K \quad \text{or} \quad \nu = \sqrt{-i}\ \sqrt{(\omega /K)}$$

which may look awkward!

The way to find the square root of a complex number $z = (x + iy)$ is to write

$$z = |z| (x/| z | + i y/| z |) = | z | \exp(i \theta) \text{ where } \theta = \arctan(y/x)$$

Then $\sqrt{z} = \pm \sqrt{|z|} \exp(i \theta/2)$.

So for $z = -i$, $x= 0$, $y = -1$, $| z | = 1$, $\theta = 3\pi/2$, and

$$\sqrt{-i} = \exp(i 3\pi/4) = \cos 3\pi/4 - i \sin 3\pi/4 = \pm (-1/\sqrt{2} + i /\sqrt{2})$$

So $\qquad \nu = \pm\sqrt{(\omega /2K)} (-1 + i)$

And our solution is

$$T(x_j, t) = \exp(- i \omega t) [X_0\exp(\pm\sqrt{(\omega /2K)} (-1 + i) x_1) + X_1\exp(- (\pm)\sqrt{(\omega /2K)} (-1 + i) x)]$$

Now to keep the solution bounded in x, choose the '+' solution for the first, and then because $x \geq 0$ the second is redundant, so $X_1 = 0$.

$$T(x , t) = X_0 \exp(- i \omega t) \exp(\sqrt{(\omega /2K)} (-1 + i) x)$$

or

$$T(x , t) = X_0 \exp(-\sqrt{(\omega /2K)} x) \exp(i [\sqrt{(\omega /2K)} x - \omega t])$$

We should recognise the second exponential: it is a plane wave travelling in the +x direction with speed = $\sqrt{(\omega 2K)}$. The wave is damped by the real exponential: $\exp(-\sqrt{(\omega /2K)} x)$.

The real part of the solution is

$$T_c(x , t) = X_c \exp(-\sqrt{(\omega /2K)} x) \cos(\sqrt{(\omega /2K)} x - \omega t)$$

We may need to allow for a phase; so in general we will need a sin term as well

$$T_s (x , t) = X_s \exp(-\sqrt{(\omega /2K)} x) \sin(\sqrt{(\omega /2K)} x - \omega t) \qquad (6b)$$

Solution to match boundary conditions: Fourier series for initial condition

The temperature on x = 0 is a periodic function $\phi(t)$. This is *necessarily smooth* since there can never be discontinuities in temperature or its gradient unless there is an interventionary 'spike'.

This is sufficient for us to expand $\phi(t)$ as a Fourier series:

$$\phi(t) = \sum_{n=0}^{\infty} A_n \cos(n\,\omega\,t) + \sum_{n=0}^{\infty} B_n \sin(n\,\omega\,t) = 0$$

Each term will separately be a solution at x = 0. The frequency in each term is $n\omega$, which we need to substitute for in (6a, b). So the full solution to the problem is

$$T(x, t) = \sum_{n=0}^{\infty} A_n \exp(-\sqrt{(n\omega/2K)}\,x) \cos(n\omega t - \sqrt{(n\omega/2K)}\,x)$$
$$+ \sum_{n=0}^{\infty} B_n \exp(-\sqrt{(n\omega/2K)}\,x) \sin(n\omega t - \sqrt{(n\omega/2K)}\,x)$$

Specific problem: What is the change of temperature in the Earth with depth if the surface temperature is given by:

$$T(0, t) = A_0 + A_1 \cos(\omega t)$$

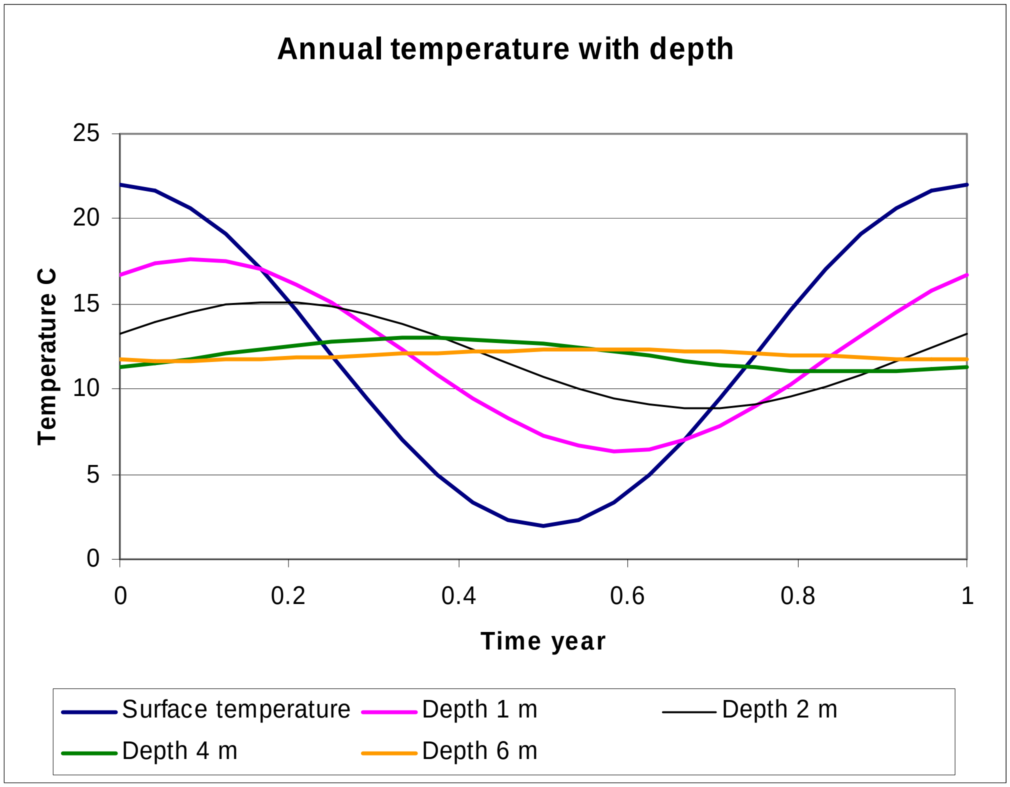
Where ω = the angular frequency corresponding to a period τ of 1 year

$= 3.15 \times 10^{7}$ s.

The answer is immediate!

$$T(x, t) = A_0 + A_1 \exp(-\sqrt{(\omega/2K)}\,x) \cos(\omega t - \sqrt{(\omega/2K)}\,x)$$

where $= 2\pi/3.15 \times 10^{7} = 2.0 \times 10^{-7}$ rad/s, and K is the thermal diffusivity of the material at the surface, usually about $1 - 10 \times 10^{-7}$ $m^2\,s^{-1}$.



Annual temperature with depth
25
20
15
10
5
0
Temperature C
0
0.2
0.4
0.6
0.8
1
Time year
Surface temperature
Depth 1 m
Depth 2 m
Depth 4 m
Depth 6 m