MATH32x PDEs Module

Assignment 1 (25%) *Due:* **Friday 10 May 2013**

Q1 A P wave potential is given by:

$$\phi(x_1, x_2, x_3, t) = \phi_0 \exp(i(k_j x_j - \omega t)), \text{ where } \phi_0, k_j \text{ and } \omega \text{ are constants.}$$

Find the displacement vector u_p and show that it is parallel to the propagation direction.

(NB do NOT change axes to make k_j parallel to x_1')

Q2a. Use the terminology and equations in the appendix to Ch 1 to find h_1, h_2, h_3, ∇f, $\nabla \bullet \underline{\mathbf{v}}$ and show that $\nabla^2 f$ for *cylindrical polar coordinates* (r, θ, z) is given by

$$\nabla^2 f = (1/r)\, \partial/\partial r\, (r\, \partial f/\partial r) + 1/r^2\, \partial^2 f/\partial\theta^2 + \partial^2 f/\partial z^2$$

NB In cylindrical polar coordinates, $\underline{\mathbf{r}} = (x, y, z) = (r\cos\theta, r\sin\theta, z)$

b. Find a separated solution for the wave equation

$$\partial^2 \phi/\partial t^2 = c^2 \nabla^2 \phi$$

where $\phi(r, z, t)$ is the potential for P wave propagating along the z axis of a uniform circular rod (no θ dependence) as follows.

(i) Show $\phi(r, z, t) = R(r)\, Z(z)\, T(t) = R(r) \exp(i[\omega t \pm \nu z])$,

where $k^2 = (\omega/c)^2 - \nu^2$ and R satisfies $(1/r)\, \partial/\partial r\, (r\, \partial R/\partial r) + k^2 R = 0$

(ii) Write $x = k r$, k constant, and show that $\partial R/\partial r = k\, \partial R/\partial x$.

(iii) Hence show that R satisfies $(1/x)\, \partial/\partial x\, (x\, \partial R/\partial x) + R = 0$.

(iv) $(1/x)\, \partial/\partial x\, (x\, \partial R/\partial x) + (1 - n^2) R = 0$ is Bessel's equation for order n with solutions $J_n(x)$, $Y_n(x)$.

Use this to show that $\phi(r, z, t) = J_0(kr) \exp(i[\omega t \pm \nu z])$.

(v) Sketch ϕ for any fixed t, z.

(vi) What boundary condition will control the choice of k?

Q3**a**. Solve (numerically) the cubic equation 8 of Chapter 1.2:

$$(c^2/\beta^2)^3 - 8(c^2/\beta^2)^2 + (24 - 16\,\beta^2/\alpha^2)(c^2/\beta^2) - 16(1 - \beta^2/\alpha^2) = 0$$

for (c^2/β^2) when $\alpha = 6$ km/s and $\beta = 3.7$ km/s.

b Hence find A/B (equation 7).

NB (1) Any method that works is acceptable e.g. Newton's method, MAPLE, MatLab etc.
(2) There are three roots to the cubic. Make sure you pick one that satisfies the constraints.

c. Use the result of Q3a, b to calculate the amplitudes $| u_1 |$, and $| u_3 |$ (i.e moduli) of Rayleigh waves at $x_1 = 0$, $t = 0$:

$$u_1 = i k A \exp(- i k (c^2 / \alpha^2 - 1)^{½} x_3)$$

$$+ i k B (c^2 / \beta^2 - 1)^{½} \exp(- i k (c^2 / \beta^2 - 1)^{½} x_3)$$

$$u_3 = - i k A(c^2 / \alpha^2 - 1)^{½} \exp(- i k (c^2 / \alpha^2 - 1)^{½} x_3)$$

$$+ i k B \exp(- i k (c^2 / \beta^2 - 1)^{½} x_3)$$

remembering that $c^2 / \beta^2 - 1 < 0$ so the arguments of the exp's are real.

(i) Calculate the functions for $k = 1, 0.5, 0.1$. Take $B = 1$. *Plot* $| u_1 |$, and $| u_3 |$ as functions of x_3 for $0 > x_3 > -50$ km (*NB x_3 is negative*) for each k.

NB Maple or Excel, is recommended for this.

(ii) What can you conclude from this about how far Rayleigh waves of different wavelengths "see" into the earth? If the wavespeeds α and β increase with depth, what effect will this have on the speeds of the Rayleigh waves with different wavenumbers k?

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Tutorial exercises for Friday 3 May.

1. Find ∇f , $\nabla \bullet \underline{v}$ and $\nabla^2 f$ in spherical polar coordinates.

2. An S wave potential is given by:

$$\underline{\psi} (x_1, x_2, x_3, t) = \underline{\psi}_0 \exp (i (k_j x_j - \omega t)$$

where $\underline{\psi}_0$ is a fixed vector. Show that the displacement u_i is perpendicular to the propagation direction.

3. A continuum consists of two uniform elastic half spaces welded together at their plane interface. The Lame constants for the two materials are:

$$\lambda, \mu , \text{ and } \lambda', \mu'$$

Find a condition linking the stress tensors in each medium, across the boundary.

4. With the geometry of Chapter 1.2 p1, show that an interface wave can exist with displacement u_2 when the continuum consists of a layer of thickness Z over a half-space: i.e. a half space M, a layer M ' of thickness Z; with P and S wave speeds and density specified in each layer.