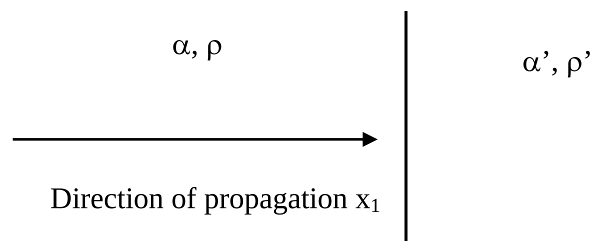
MATH/GPHS 322/323 DEs module

Assignment 4 (25%) Due 5pm Friday 14 June 2013

Question 1.

A plane compressional wave is propagation through an isotropic, homogenous space with speed . It arrives at a plane boundary to another isotropic, homogenous medium with speed α' and density ρ', where the normal to the boundary is the direction of wave propagation. A wave is reflected from the boundary and a wave is transmitted through the boundary.



The potential for the incident wave can be written:

$$\phi_{\,in} = A \exp\left(i\,[t - x_1/\alpha\,]\right)$$

(i) If the amplitudes of the reflected and transmitted potentials are A_r and A', write down the potentials for the transmitted and reflected waves.

(ii) What are the boundary conditions that the waves have to satisfy?

(iii) For each medium, Hooke’s law can be written:

$$S_{i\,j} = \lambda\, \delta_{i\,j}\, \partial u_k/\partial x_k + \mu\,(\partial u_i/\partial x_j + \partial u_j/\partial x_i\,)$$ with λ' and μ' in the RH medium.

Show that Hooke’s law simplifies to

$$S_{11} = (\lambda + 2\,\mu)\, \partial u_1/\partial x_1$$

Note: $(\lambda + 2\,\mu) = \rho\,\alpha^2$

(iv) Hence show that the ratios A_r/A and A'/A are given by

$$A_r/A = (\rho'\alpha' - \rho\,\alpha\,)\,/\,(\rho'\alpha' + \rho\,\alpha\,)$$

$$A'/A = 2\,\rho\,\alpha'\,/\,(\rho'\alpha' + \rho\,\alpha\,)$$

(v) What are A_r and A' if the Right Hand medium is a vacuum?

Question 2. A solution to Laplace's equation in the quarter space $x \geq 0$, $y \geq 0$ is given by

$$u_z(x,y) = (U/\pi)\,[\,\arctan((b - x)/y) + \arctan((b + x)/y) - \arctan((a - x)/y) - \arctan((a + x)/y))\,]\,; \quad y \neq 0$$

(a) Write down the limit as $y \to 0^+$ of $\arctan((a - x)/y)$ for $x > a$.

(b) Hence find the limit as $y \to 0^+$ of $u_z(x,y)$ for

(i) $a < x < b$,

(ii) $x > b$ and

(iii) $x < a$

(c) Verify that the shear stress S_{xz} across $x = 0$ is zero.

(d) Hence give an interpretation for $u_z(x,y)$.

Question 3. Two particular solutions to the heat diffusion equation are given by.

(1) $$T(x, t) = A\,\mathrm{erfc}(x\,(4Kt)^{-1/2}) = A\left\{1 - 2/\sqrt{\pi}\int_0^{x(4Kt)^{-1/2}} \exp(-\xi^2)\,d\xi\right\}$$

in the half space $x \geq 0$, $t \geq 0$

(2) $$T(x, t) = \int_{-\infty}^{\infty} F(\omega)\exp(i\,\omega\,x)\exp(-\omega^2 K\,t)\,d\omega$$

for $-\infty \leq x \leq -\infty$, $t \geq 0$

(a) Show that they satisfy the heat diffusion equation $\partial T/\partial t - K\nabla^2 T = 0$.

(b) By considering the results of taking appropriate limits for x and t, describe the circumstances, or physical problem, for which they could provide a solution.

Tutorial exercises for Friday 7 June.

1. A plane P wave of amplitude A is incident obliquely at the earth's surface, at incident angle θ.

Use the method of potentials to prove that the angle of reflection = angle of incidence, and Snell's law.

2. If $\theta = 0$, show that the amplitude at the surface is 2A.

3. A plane SH wave of amplitude B is incident at $\theta = 0$ on a plane interface between homogeneous isotropic materials with shear wavespeeds and densities of β,ρ and β',ρ' respectively. Find the ratio of the transmitted wave amplitude to B.