MATH/GPHS 322/323 DEs Module

Assignment 3 (25%) ***Due:*** **Friday 31 May**

1. For the ocean crust cooling problem, the temperature at a depth x at time t is given by:

$$T(x, t) = A \operatorname{erf}(x\,(4Kt)^{-1/2})$$

Develop a model for the evolution of the oceanic lithosphere (crust plus upper mantle) as follows.

(i) First note that T(x, t) = (constant temperature) for $x\,(4Kt)^{-1/2}$ = constant. Use NORM.S.INV function in Excel (or equivalent elsewhere, e.g. Maple, MatLab) to find the values of $\xi = x\,(4Kt)^{-1/2}$ for a range of temperatures from T= 0C to 1200C. On one graph, plot ξ v. T for each of these.

NB take $K = 1 \times 10^{-6}$ m^2/s

(ii) A point P_0 is at a mid-ocean ridge at t = 0. The surface temperature *A* is 1200C.

Assume that P_0 moves away from the ridge crest at a steady speed dy/dt of 50mm/yr = 50 km/My. This enables the conversion of time into distance y from the ridge by y = 50 t.

Use the results of part (i) to find the depth x(t) for the following isotherms (T = constant):

I. T = 200 C;

II. T = 580 C, corresponding to the Curie point for the mineral magnetite. At higher temperatures than the Curie point, the rock is too hot to maintain a magnetic field induced by the Earth's magnetic field.

III. T = 1100 C, corresponding to the (nominal) transition from lithosphere to asthenosphere.

Use the result from lectures that the subsidence C(t) of the sea-floor is given by

$$C(t) = (1.45*230)\, t^{1/2} \text{ m, with t in Myear}$$

to **plot a profile of the ocean lithospheric structure** between x = 0 to 200 km depth, for times t = 0 to 100 My, showing the depth of the ocean floor and the isotherms I, II and III.

That is, plot the subsidence v. distance, and the isotherms, each isotherm being subsided by C(t).

What would you predict the thickness of 100 My old oceanic lithosphere to be?

2. (a) Show that:

$$T(x, t) = \int_{-\infty}^{\infty} (4\pi K t)^{-1/2} f(\xi) \exp[- (x-\xi)^2/(4K t)] \, d\xi$$

is a solution of the heat diffusion equation.

(b) Change the variable to $\zeta^2 = (x-\xi)^2/(4K t)$ and take the limit as $t \to 0$.

(c) Hence find the solution to the problem: Find the temperature in an infinite solid when the temperature at $t = 0$ is $T(x, 0) = f(x)$.

3. Solve the problem of diurnal (daily) temperature variation in the Earth. Assume minimum night-time temperatures are on average 12C cooler than maximum daytime temperature. Plot the temperature change with depth in the Earth starting from a maximum at the surface. If temperature can be measured to = 0.05C, at what depth is the diurnal change insignificant? Hence explain why the diurnal change can be neglected when modeling the seasonal change.

Hint: Program the solution for a general frequency (and depth increment) which can then be used for either the diurnal or annual problems. *Be careful* with the units of time.

4. Thermistors are buried at depths of 2 and 4 metres. If the maximum temperature difference between the two thermistors over a year is 2.4C, and the annual temperature range is ± 10C, estimate the thermal diffusivity of the soil as follows.

(a) Write down an expression for the temperature difference between 2 and 4 metres, T(2,t)-T(4,t), as a function of time and differentiate this to find an equation for the time at which the maximum (or minimum) occurs.

(b) Show that this expression, using the notation of lectures, can be simplified to

$$\omega t - 2\sqrt{(\omega /2K)} = \operatorname{atan}\{ \sin(2\sqrt{(\omega /2K)})/[\cos(2\sqrt{(\omega /2K)}) - \exp(2\sqrt{(\omega /2K)})] \}$$

(c) Now substitute this value for $\omega t - 2\sqrt{(\omega /2K)}$ into T(2,t)-T(4,t) and plot T(2,t)-T(4,t) versus K for a range of values of K,

(d) Read off the value of K matching T(2,t)-T(4,t) = 2.4C.

NB Sin(x + a) =sin x cos a + cos x sin a.

Tutorial exercises for Friday 24 and 31 May

1. Establish the trig identities for cos and sin of the sum and difference of angles A and B by considering the successive rotations of the x_1 and x_2 coordinates axes about x_3, through first A then B.

2. Show that

$$\int_{-\infty}^{\infty} 1/\sqrt{(\pi)}\ \exp[-\xi^2]\ d\xi = 1$$

3. Find the nth root of 1 + i

4. Excel tutorial on erf(x)

5. Find the equation of the 900C isotherm beneath a cooling ocean that is spreading from a ridge at 30 mm/yr.

6. (a) Find the maxima and minima of temperature with depth in the seasonal temperature change problem.

   (b) For the seasonal temperature change problem, suppose it is mid-summer: find the depth of the ‘last winter’.

Using Excel to calculate erf(x)

(1) Go back to the equations:

$$\mathrm{erf}(y) = \sqrt{(2/\pi)} \int_0^{\sqrt{2}y} \sqrt{2} \exp(-\zeta^2/2)\, d\zeta / \sqrt{2} = 2/\sqrt{\pi} \int_0^{y} \exp(-\xi^2)\, d\xi$$

Excel does not have an erf() function but it has the (equivalent) cumulative Gaussian distribution NORM.S.DIST(), where

$$\mathrm{NORM.S.DIST}(y) = 1/\sqrt{(2\pi)} \int_{-\infty}^{y} \exp(-\zeta^2/2)\, d\zeta$$

Now

$$\mathrm{NORM.S.DIST}(0) = 1/\sqrt{(2\pi)} \int_{-\infty}^{0} \exp(-\zeta^2/2)\, d\zeta = 0.5$$

So $\mathrm{erf}(y) = 2*(\mathrm{NORM.S.DIST}(\sqrt{2}y) - 0.5)$. (A1)

(The ‘S’ in the middle is for ‘standard’, meaning that the mean = 0, standard deviation = 1).

(2) The problem of finding the value of y that corresponds to a value of NORM.S.DIST(y) is solved using NORM.S.INV.

e.g. find y such that erf(y) = 0.5.

then by A1, NORM.S.DIST($\sqrt{2}$y) = 0.5+erf(y)/2 = 0.5+0.25 =0.75

NORM.S.INV(0.75) = 0.6745, so y = 0.6745/sqrt(2) = 0.4769

(3) FOR GEOLOGICAL TIME/DISTANCE SCALES: t in million years (Myr) and x in km. Note that K is in SI units of $m^2\ s^{-1}$.

We will need to calculate erf(x $(4Kt)^{-1/2}$), with x in km, t in Myr and K ~ $10^{-6}\ m^2\ s^{-1}$. Covert K to $km^2\ My^{-1}$:

1year = 365.25x24x3600 sec = 31557600 sec, so 1 sec =3.1688×10^{-14} Myr.
1 $m^2 = 10^{-6}\ km^2$.

So K = $10^{-6} \times 10^{-6}$ / 3.1688×10^{-14} $km^2\ My^{-1}$

= 31.5576 $km^2\ My^{-1}$