Math/Gphs 322/323 DEs for Earth and Physical Sciences Module

Chapter 1: The wave equation

Chapter 2: Waves on an interface or surface

Chapter 1 The wave equation

P and S waves

In 'Tensors' we showed that a disturbance in a continuum can propagate stress and strain changes according to Newton's Law (via Navier's equation),

$$\rho \partial^2 u_i / \partial t^2 = \mu \partial^2 u_i / \partial x_j \partial x_j + (\mu + \lambda) \partial^2 u_k / \partial x_k \partial x_i$$

through waves that travel at two different speeds:

(1) longitudinal waves that travel with a speed given by

$$\alpha = \sqrt{\{(2 \mu + \lambda) / \rho\}}$$

The Bulk Modulus of the continuum, $\kappa = \lambda + 2/3\ \mu$; so equivalently:

$$\alpha = \sqrt{\{(\kappa + 4/3\ \mu) / \rho\}}$$

(2) shear waves, in which the displacement is at right angles to the propagation direction, and whose speed is given by:

$$\beta = \sqrt{\{\mu / \rho\}}$$

Since κ is positive, $\alpha > \beta$, the compressional waves arrive first from a source (earthquake) (hence their old name of Primary – P – waves), and the shear waves arrive later (old name Secondary – S – waves).

For glass (see Table 1), $\mu = 2.72 \times 10^{10}$ N/m^2, $\rho \approx 2$ Mg/m^3, and $\kappa = 4.5 \times 10^{10}$ N/m^2

So: $\beta = 3.7$ km/s, and $\alpha = 6.4$ km/s

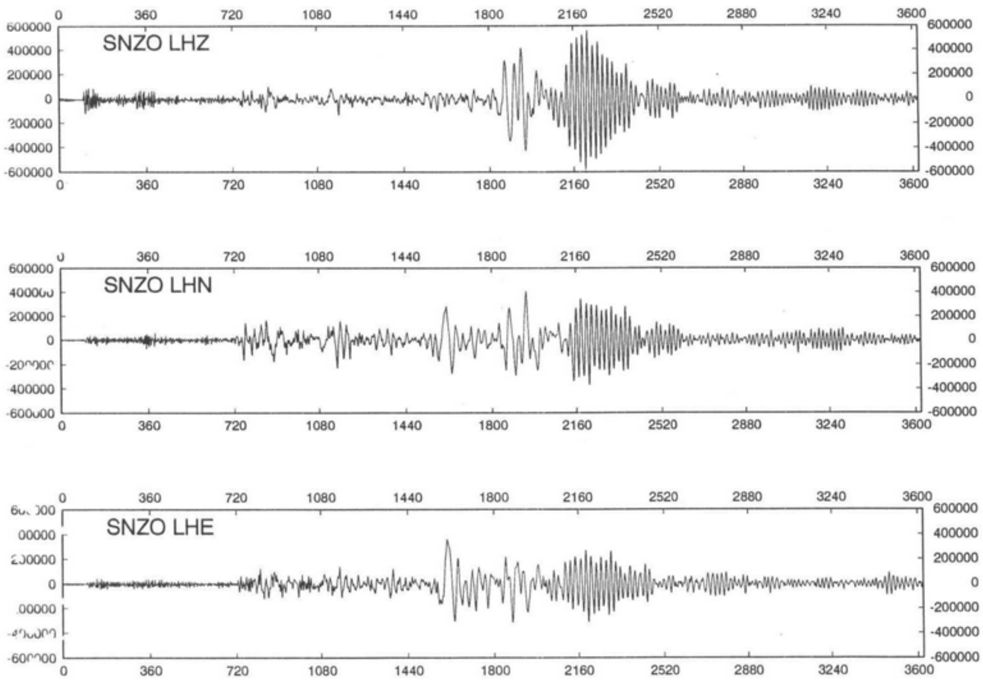
(These are typical wavespeeds in the Earth's lower crust).

Table 1 Some elastic moduli

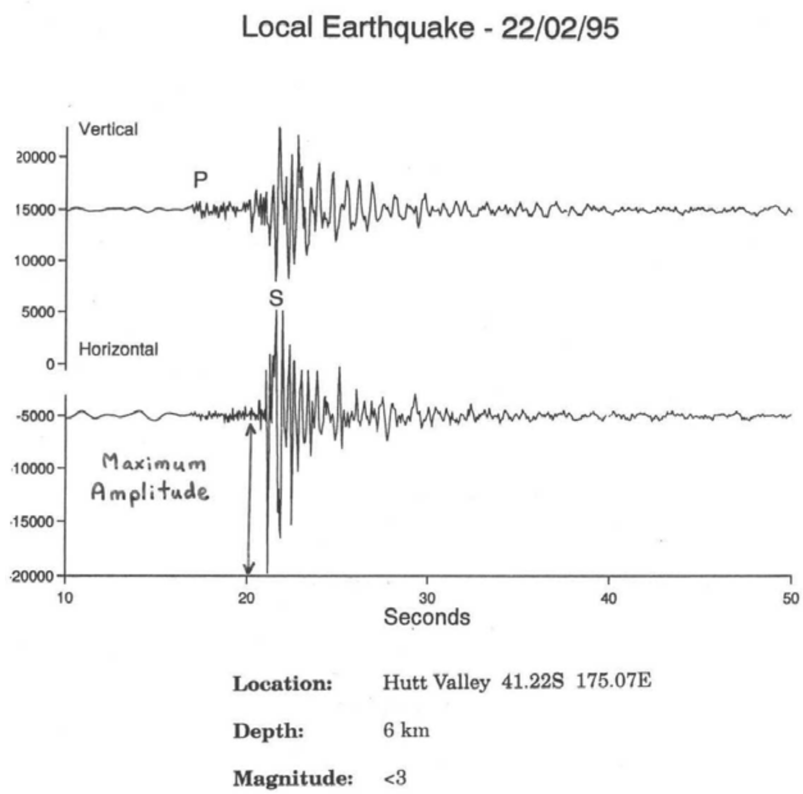
Material	Poisson's Ratio ν	λ	μ	Bulk modulus κ	Young's Modulus Y
		10^{10} N/m^2	10^{10} N/m^2	10^{10} N/m^2	10^{10} N/m^2
Steel	0.26	8.84	8.19	14.3	20.6
Gold	0.42	14.7	2.80	16.6	7.95
Copper	0.33	8.65	4.58	11.7	12.2
Glass	0.25	2.69	2.72	4.5	6.8
Fluids	0.5	Large	0	Large	0
Earth's crust (av)	0.28	4.5	3.6	6.0	9.2
Incompressible	0.5	∞	μ	∞	3 μ
Poisson's case	0.25	λ	λ	5/3 λ	5/2 λ
General	$-1 \le \nu \le 1/2$		≥ 0	≥ 0	≥ 0

For many materials, including some rocks, $\mu \approx \lambda$. Materials for which this is true are called "*Poisson solids*". In this case $\nu = 0.25$, and

$$\alpha / \beta \approx \sqrt{(2\mu + \mu) / \mu} = \sqrt{3} = 1.732\ldots$$



Broad-band (wide frequency band) seismograms from a distant earthquake recorded at Makara, Wellington (SNZO).



Two components (vertical and one horizontal) of ground motion from a small earthquake under Wellington. Note the relative sizes of the P and S waves.

Separation of Navier's Equation into Wave Equations

In general, a disturbance will have components of displacement u_j in all three coordinate directions. However, because P waves travel faster, we can separate P and S waves and follow their propagation separately.

A vector identity (see any book on vector calculus) is:

$$\nabla \mathbf{x} \nabla \mathbf{x} \; \underline{\mathbf{u}} = \nabla (\nabla \bullet \underline{\mathbf{u}}) - \nabla^2 \underline{\mathbf{u}}$$

or in tensor notation:

$$\varepsilon_{ijk} \, \partial / \partial x_j (\varepsilon_{kpq} \partial u_q / \partial x_p) = \partial^2 u_k / \partial x_k \partial x_i - \partial^2 u_i / \partial x_j \partial x_j$$

We shall use vector notation in this section as this makes it more compact to write 'curls' and "**x**" product.

Substitute for $\nabla^2 \underline{\mathbf{u}}$ in Navier's Equation:

$$\rho \, \partial^2 \underline{\mathbf{u}} / \partial t^2 = \mu \nabla^2 \underline{\mathbf{u}} + (\mu + \lambda) \; \nabla \nabla \bullet \underline{\mathbf{u}}$$

$$= \mu (\nabla (\nabla \bullet \underline{\mathbf{u}}) - \nabla \mathbf{x} \nabla \mathbf{x} \; \underline{\mathbf{u}}) + (\mu + \lambda) \; \nabla \nabla \bullet \underline{\mathbf{u}}$$

ie

$$\rho \, \partial^2 \underline{\mathbf{u}} / \partial t^2 = (2\mu + \lambda) \; \nabla \nabla \bullet \underline{\mathbf{u}} - \mu \nabla \mathbf{x} \nabla \mathbf{x} \; \underline{\mathbf{u}} \qquad (1)$$

Now we need *Helmholtz's Theorem*, which says that "nice" vector fields $\underline{\mathbf{u}}$ (i.e. ones that are differentiable everywhere, and $\rightarrow 0$ as $R \rightarrow \infty$) can be written as:

$$\underline{\mathbf{u}} = \nabla \phi + \nabla \mathbf{x} \underline{\psi} \qquad (2)$$

The functions ϕ and $\underline{\psi}$ are called scalar and vector potentials respectively. $\nabla \phi$ is *conservative*, in the sense that where $\nabla \phi$ represents a force, work done against the force travelling around a closed curve is zero. Since $\underline{\psi}$ only has two independent components (to make three for $\underline{\mathbf{u}}$) $\underline{\psi}$ can be taken to be divergence free: $\nabla \bullet \underline{\psi} = 0$.

We can use this representation for the disturbance $\underline{\mathbf{u}}$ in the continuum because it is small and it will decay away from a point source (through geometric spreading) as 1/R. Then:

$$\nabla \bullet \underline{\mathbf{u}} = \nabla \bullet \nabla \phi + \nabla \bullet \nabla \mathbf{x} \underline{\psi}$$

$$= \nabla^2 \phi \qquad \text{(second term identically zero)}$$

and:

$$\nabla \mathbf{x} \underline{\mathbf{u}} = \nabla \mathbf{x} \nabla \phi + \nabla \mathbf{x} \nabla \mathbf{x} \underline{\psi}$$

$$= \nabla \mathbf{x} \nabla \mathbf{x} \underline{\psi} \qquad \text{(first term identically zero)}$$

Therefore:

$$\rho \partial^2 \underline{\mathbf{u}} / \partial t^2 = \rho \partial^2 (\nabla \phi + \nabla \mathbf{x} \underline{\psi}) / \partial t^2 = \rho \nabla \partial^2 \phi / \partial t^2 + \rho \nabla \mathbf{x} \partial^2 \underline{\psi} / \partial t^2$$

$$= (2 \mu + \lambda) \nabla (\nabla^2 \phi) - \mu \nabla \mathbf{x} (\nabla \mathbf{x} \nabla \mathbf{x} \underline{\psi}) \qquad \text{(as (1) above)}$$

Therefore:

$$\nabla \{\rho \partial^2 \phi / \partial t^2 - (2 \mu + \lambda) \nabla^2 \phi\} + \nabla \mathbf{x} \{\rho \partial^2 \underline{\psi} / \partial t^2 + \mu \nabla \mathbf{x} \nabla \mathbf{x} \underline{\psi}\} = \underline{\mathbf{0}}$$

everywhere.

But this is the form of Helmholtz's Equation for a zero field, which can only be satisfied if the scalar and vector potentials are both zero. Ie:

$$\rho \partial^2 \phi / \partial t^2 - (2 \mu + \lambda) \nabla^2 \phi = 0$$

$$\rho \partial^2 \underline{\psi} / \partial t^2 + \mu \nabla \mathbf{x} \nabla \mathbf{x} \underline{\psi} = 0$$

Substituting

$$\alpha = \sqrt{\{(2 \mu + \lambda) / \rho\}}, \quad \beta = \sqrt{(\mu / \rho)}$$

Gives:

$$\partial^2 \phi / \partial t^2 = \alpha^2 \nabla^2 \phi$$

$$\partial^2 \underline{\psi} / \partial t^2 = - \beta^2 \nabla \mathbf{x} \nabla \mathbf{x} \underline{\psi}$$

Now use:

$$\nabla \mathbf{x} \nabla \mathbf{x} \underline{\psi} = \nabla (\nabla \bullet \underline{\psi}) - \nabla^2 \underline{\psi}$$

and remembering that $\nabla \bullet \underline{\psi} = 0$ (Helmholtz) we have:

$$\partial^2 \phi / \partial t^2 = \alpha^2 \nabla^2 \phi$$

$$\partial^2 \underline{\psi} / \partial t^2 = \beta^2 \nabla^2 \underline{\psi} \qquad (3)$$

which are the *wave equations* for longitudinal and shear waves respectively.

Note that we appear to have exchanged 3 unknowns u_j for 4: ϕ and ψ_j. However, we have $\nabla \bullet \underline{\psi} = 0$ which means that only two of ψ_j are independent.

The displacements can be recovered using eqn (2): take ∇ of 3a and ∇ **x** of 3b:

$$\partial^2 (\nabla\phi) / \partial t^2 = \alpha^2 \nabla^2 (\nabla\phi)$$

$$\partial^2 (\nabla \times \underline{\psi}) / \partial t^2 = \beta^2 \nabla^2 (\nabla \times \underline{\psi}) \qquad (3^*)$$

In this formulation we have decoupled the P and S parts of the solution. *This cannot be done in general for anisotropic materials.*

Wave equation in other coordinate systems

The form of the wave equation:

$$\partial^2 f / \partial t^2 = c^2 \nabla^2 f$$

means that we can calculate solution to the wave equation in other coordinates if we write down $\nabla^2 = \nabla \bullet \nabla$ in them; for example, in spherical polar coordinates.

Spherical Polar Coordinates

$\nabla^2 f$ in spherical polars (see Appendix) is

$$\nabla \bullet \nabla f = (1 / r^2)\, \partial / \partial r\, (r^2 \partial f / \partial r) + (1 / r^2 \sin\theta)\, \partial / \partial\theta\, (\sin\theta\ \partial f / \partial\theta)$$

$$+ (1 / r^2 \sin^2\theta)\, \partial^2 f / \partial^2 \phi$$

Wave equation in spherical polars – r dependence only

For a function depending only on r, the wave equation becomes:

$$\partial^2 f / \partial t^2 - c^2 \nabla^2 f = \partial^2 f / \partial t^2 - c^2 (1 / r^2)\, \partial / \partial r\, (r^2 \partial f / \partial r) = 0$$

Try a solution of the form (1/r) f(r - ct), where f is suitably differentiable. LHS is:

$$c^2 (1/r) f'' - c^2 (1 / r^2)\, \partial / \partial r\, (r^2 \partial (f/r) / \partial r)$$

$$= c^2 (1/r) f'' - c^2 (1 / r^2)\, \partial / \partial r\, (r^2 f' / r - r^2 f / r^2)$$

$$= c^2 (1/r) f'' - c^2 (1 / r^2)\, \partial / \partial r\, (r f' - f)$$

$$= c^2 (1/r) f'' - c^2 (1 / r^2) (r f'' + f' - f')$$

$= c^2 (1/r) f'' - c^2 (1/r) f''$

$= 0$

So this is a solution. The difference from the plane wave solutions in Cartesian coordinates is, of course, that as r increases the pulse is diminished by (1/r).

Fourier representation of the wave pulse

Earthquakes generate waves of different frequencies, which may be identified by Fourier analysis of the waves. We can transform the wave pulse f(t) recorded at a station:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(i \omega t) \, dt$$

In practice, the limits are the duration of the waves, or part thereof. We can then recover the waveform with the inverse transform:

$$f(t) = (1/2\pi) \int_{-\infty}^{\infty} F(\omega) \exp(-i \omega t) \, d\omega$$

We can think of the contribution to the wave at (angular) frequency ω ($= 2\pi\nu$; ν is 'ordinary' frequency in Hz) as having (complex) amplitude $A = F(\omega)d\omega$, which multiplies a harmonic (sine plus cosine) function $\exp(-i \omega t)$.

We can convert this from a seismogram to the equation of a plane wave by replacing ωt with $2\pi(x/L \pm \nu t)$; viz. $\exp(-i\, 2\pi(x/L \pm \nu t))$. that is we have a solution

$$f(2\pi(x/L \pm \nu t)) = (1/2\pi) \int_{-\infty}^{\infty} F(\omega) \exp(-i\, 2\pi(x/L \pm \nu t)) \, d\omega$$

It will therefore suffice, and be convenient from now on, to only consider harmonic waves, i.e. solution to Navier's equation of the form $\exp(-i\, 2\pi(x/L \pm \nu t))$ (or use sin and cos, without the complex i) since we can build an arbitrary wave by summing these up, as required.

We can multiply (or divide) the argument ($x / L \pm \nu t$) by a constant without affecting the function being a solution. So if $g(x / L \pm \nu t)$ is a solution, so is

$$g(x \pm \nu L t) = g(x \pm c t).$$

where c is the wavespeed, β or α.

Nomenclature. L is the wavelength, and $2\pi / L$ is called the *wavenumber*, often denoted k. We can have a vector wavenumber **k**, in which case (xk) in the argument of f is replaced by
$x_j k_j = \mathbf{x} \cdot \mathbf{k}$ (see 'Tensors').

These solutions are *plane waves*. All points in the plane $x_j k_j$ = constant
have the same value of $f(x_j k_j \pm \omega t)$, and so these points constitute a *plane wave front*, propagating in the direction of **k** , with speed $= \omega / | \mathbf{k} |$.

Plane waves revisited: separated solutions of the Wave Equation

We shall now show how plane wave solutions arise as particular solutions to the Wave equation in Cartesian coordinates.

We can derive a solution for the wave equations (3) that turns out to be plane waves. We will try to find a solution to the compressional wave equation:

$$\partial^2 \phi / \partial t^2 = \alpha^2 \nabla^2 \phi$$

of the form:

$$\phi(x_1, x_2, x_3, t) = X(x_1)Y(x_2)Z(x_3)T(t) \qquad (4)$$

This method of seeking a solution is called *separation of variables* – for obvious reasons. It is *not* guaranteed to work in any particular problem! Whether we can find such a solution will depend on the boundary conditions. It is however a good option to try when we have planar boundaries. We can then chose one of the coordinate axes to be normal to one of the boundaries.

Notice that a soluton of this kind reduces the PDE (3) to a set of ordinary differential equations.

Substituting ϕ into:

$$\partial^2 \phi / \partial t^2 - \alpha^2 \nabla^2 \phi = 0$$

gives:

$$X(x_1)Y(x_2)Z(x_3)\, d^2T(t)/dt^2 - \alpha^2 (d^2X(x_1)/dx_1^2\, Y(x_2)Z(x_3)\, T(t)$$

$$+ X(x_1)\, d^2Y(x_2)/dx_2^2\, Z(x_3)\, T(t) + X(x_1)\, Y(x_2)\, d^2Z(x_3)/dx_3^2\, T(t)) = 0$$

Divide through by $X(x_1)Y(x_2)Z(x_3)T(t)$. This will be allowed because we do not want any of X, Y, Z, T to be zero everywhere. We get:

$$(1/T)\, d^2T(t)/dt^2 - \alpha^2 ((1/X)\, d^2X(x_1)/dx_1^2 + (1/Y)\, d^2Y(x_2)/dx_2^2 + (1/Z)\, d^2Z(x_3)/dx_3^2) = 0 \qquad \ldots\ldots\ldots\ldots\ldots..(5)$$

This means that:

$(1/T)\, d^2T(t)/dt^2$ and the three terms like $(1/X)\, d^2X(x_1)/dx_1^2$

must be *constant* (to see this differentiate eqn (5) with respect to t or x_j).

So put:

$$(1/T)\, d^2T(t)/dt^2 = -\omega^2, \text{ i.e. } d^2T(t)/dt^2 + \omega^2 T = 0,$$

and

$$(1/X)\, d^2X(x_1)/dx_1^2 = -k_1^2, \text{ i.e. } d^2X(x_1)/dx_1^2 + k_1^2 X = 0 \text{ ; similarly for Y, Z.}$$

So we have solutions to these ODEs:

$$T = A \exp(\pm i \omega t),$$

and

$$X = X_0 \exp(\pm i\, k_1 x_1),$$

$$Y = Y_0 \exp(\pm i\, k_2 x_2),$$

$$Z = Z_0 \exp(\pm i\ k_3 x_3),$$

where A, X_0, Y_0, Z_0 are constants. Substitute into equation (5):

$$(1/T)\, d^2 T(t)/dt^2 - \alpha^2 ((1/X)\, d^2 X(x_1) / dx_1^2 + (1/Y)\, d^2 Y(x_2) / dx_2^2 + (1/Z)\, d^2 Z(x_3) / dx_3^2)$$

$$= - \omega^2 - \alpha^2 (- k_1^2 - k_2^2 - k_3^2) = 0$$

or

$$\omega^2 = \alpha^2 (k_1^2 + k_2^2 + k_3^2) \qquad (6)$$

or $\alpha = \omega / |\, \mathbf{\underline{k}} \,|$, as before.

Thus we have

$$\phi (x_1, x_2, x_3, t) = X(x_1)Y(x_2)Z(x_3)T(t) = \phi_0 \exp (\pm i (k_j x_j \pm \omega t) \qquad (7)$$

(ϕ_0 is a constant = A X_0 Y_0 Z_0) subject to the constraint (6). In practice, for given wavespeed α and frequency ω, this constrains one of the k_j, viz:

$$k_3^2 = \omega^2/\alpha^2 - k_1^2 - k_2^2 \qquad (6^*)$$

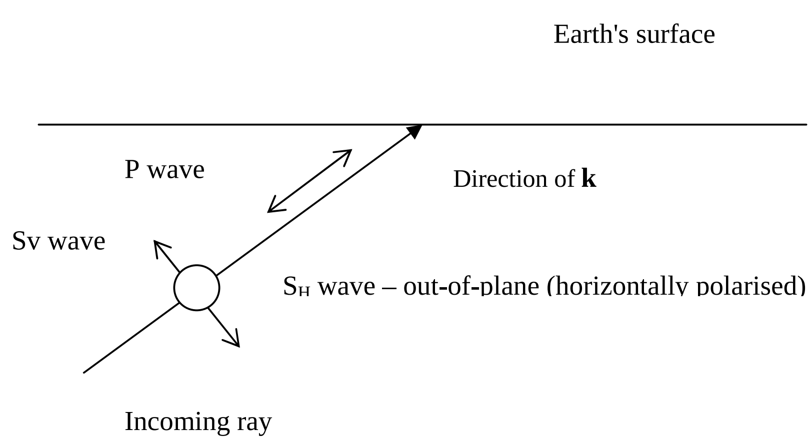
These are harmonic plane waves, as already discussed. There is no variation of ϕ in directions at right angles to $\mathbf{\underline{k}}$. Let $\mathbf{\underline{x}}$ be such a point, so that $\mathbf{\underline{x}} \cdot \mathbf{\underline{k}} = 0$. Then

$$\phi (x_1, x_2, x_3, t) = \phi_0 \exp (\pm i (0 \pm \omega t)$$

That is, ϕ is the same everywhere an any time t. **k** is normal to the wavefront, and so defines the direction of propagation of the wave. | **k** | is the wavenumber = 2π / wavelength.

Standard polarisations: P, S_V and S_H

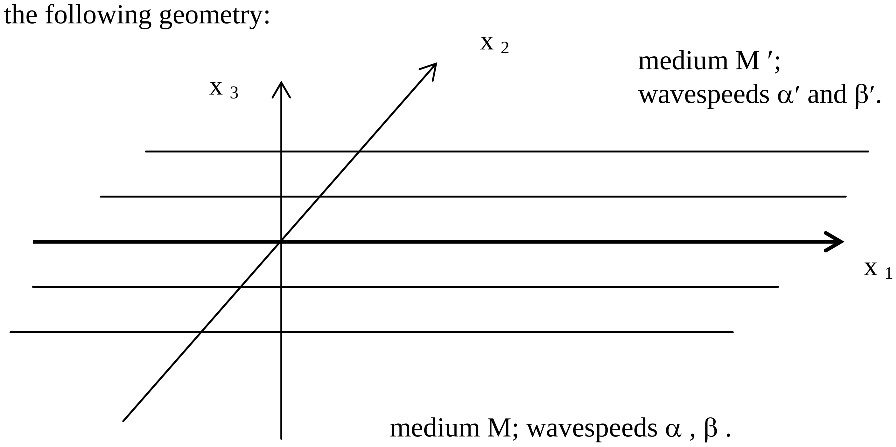
Equations (3) divide the waves into P and S. There are two independent S waves. It is usual and convenient to take two specific independent components to describe the S waves: one polarised in a vertical plane, S_V , and the other horizontal, S_H .



Chapter 2: Waves on an interface or surface

Apart from the P and S waves we have already met, there is another class of waves in continuous elastic media which are important: waves propagating along an interface between two regions, or on the surface of a region. In particular, waves from shallow earthquakes propagate around the surface of the Earth.

Consider the following geometry:



Consider a wave traveling in the x_1 direction so that:

- The disturbance is largely confined to the neighbourhood of the boundary between M and M ′; and
- It is like a plane wave in that at any time all points on any line parallel to the x_2 axis have equal displacements. NB there may be displacements in the x_2 direction.

From the latter, all derivatives with respect to x_2 will be zero.

Therefore we can replace the vector potential $\underline{\boldsymbol{\psi}}$ with a scalar ψ. The displacement due to $\underline{\boldsymbol{\psi}}$ is given by:

$$\underline{\mathbf{u}} = \nabla \mathbf{x} \ \underline{\boldsymbol{\psi}}$$

i.e.

$$u_1 = \partial \psi_3 / \partial x_2 - \partial \psi_2 / \partial x_3$$

$$u_2 = \partial \psi_1 / \partial x_3 - \partial \psi_3 / \partial x_1$$

$$u_3 = \partial \psi_2 / \partial x_1 - \partial \psi_1 / \partial x_2$$

(and $\partial \psi_1 / \partial x_1 + \partial \psi_2 / \partial x_2 + \partial \psi_3 / \partial x_3 = 0$)

But for the displacements in the 1 and 3 directions there is no variation in the 2 direction; so these reduce to:

$$u_1 = - \partial \psi_2 / \partial x_3$$

$$u_3 = \partial \psi_2 / \partial x_1$$

So we only need $\psi = \psi_2$.
So in place of

$$\partial^2 \underline{\boldsymbol{\psi}} / \partial t^2 = \beta^2 \nabla^2 \underline{\boldsymbol{\psi}} \qquad (3^*)$$

we have, in this case, the scalar wave equation

$$\partial^2 \psi / \partial t^2 = \beta^2 \nabla^2 \psi \qquad (3^{**})$$

Therefore, we can describe the motion using two scalar potentials, ϕ and ψ; and as before the displacements are:

$$u_1 = \partial \phi / \partial x_1 - \partial \psi / \partial x_3$$

$$u_3 = \partial \phi / \partial x_3 + \partial \psi / \partial x_1 \qquad (1)$$

Hence:

$$\nabla^2 \phi = \partial^2 \phi / \partial x_1{}^2 + \partial^2 \phi / \partial x_3{}^2$$

$$= \partial u_1 / \partial x_1 + \partial^2 \psi / \partial x_3 \partial x_1 + \partial u_3 / \partial x_3 - \partial^2 \psi / \partial x_3 \partial x_1$$

$$= \partial u_1 / \partial x_1 + \partial u_3 / \partial x_3 = \text{the dilatation } \theta \qquad (2a)$$

and

$$\nabla^2 \psi = \partial^2 \psi / \partial x_1{}^2 + \partial^2 \psi / \partial x_3{}^2$$

$$= \partial u_3 / \partial x_1 - \partial^2 \phi / \partial x_3 \partial x_1 - \partial u_1 / \partial x_3 + \partial^2 \phi / \partial x_3 \partial x_1$$

$$= \partial u_3 / \partial x_1 - \partial u_1 / \partial x_3 \qquad (2b)$$

The potentials, and any displacement u_2 will satisfy:

$$\partial^2 \phi / \partial t^2 = \alpha^2 \nabla^2 \phi$$

$$\partial^2 \psi / \partial t^2 = \beta^2 \nabla^2 \psi$$

$$\partial^2 u_2 / \partial t^2 = \beta^2 \nabla^2 u_2 \qquad \text{(from Navier's equation)} \qquad (3)$$

We now look for a solution of the form:

$$\phi = f(x_3) \exp[\, i k (x_1 - c t)]$$

$$\psi = g(x_3) \exp[\, i k (x_1 - c t)]$$

$$u_2 = h(x_3) \exp[\, i k (x_1 - c t)]$$

and similar relations in medium M':

$$\phi' = f(x_3)' \exp[\, i k (x_1 - c t)]$$

$$\psi' = g(x_3)' \exp[\, i k (x_1 - c t)]$$

$$u_2' = h(x_3)' \exp[\, i k (x_1 - c t)] \qquad (4)$$

NB $f(x_3)'$ does NOT mean df/dx_3 !

Substitute trial solutions (4) into (3) e.g. for ψ :

$$\partial \psi / \partial x_1 = i k g(x_3) \exp[i k (x_1 - c t)]$$

$$\partial^2 \psi / \partial x_1^2 = - k^2 g(x_3) \exp[i k (x_1 - c t)]$$

$$\partial \psi / \partial x_3 = d g(x_3) / dx_3 \exp[i k (x_1 - c t)]$$

$$\partial^2 \psi / \partial x_3^2 = d^2 g(x_3) / dx_3^2 \exp[i k (x_1 - c t)]$$

$$\partial \psi / \partial t = - i k c g(x_3) \exp[i k (x_1 - c t)]$$

$$\partial^2 \psi / \partial t^2 = - k^2 c^2 g(x_3) \exp[i k (x_1 - c t)]$$

so:

$$\partial^2 \psi / \partial t^2 - \beta^2 \nabla^2 \psi = - k^2 c^2 g(x_3) \exp[i k (x_1 - c t)]$$

$$- \beta^2 \{ - k^2 g(x_3) \exp[i k (x_1 - c t)] + d^2 g(x_3) / dx_3^2 \exp[i k (x_1 - c t)] \}$$

$$= 0$$

iff

$$- k^2 c^2 g(x_3) - \beta^2 \{ - k^2 g(x_3) + d^2 g(x_3) / dx_3^2 \} = 0$$

i.e.

$$d^2 g(x_3) / dx_3^2 + k^2 g(x_3) [c^2 / \beta^2 - 1] = 0$$

which has a solution:

$$g(x_3) = B \exp(- i k [c^2 / \beta^2 - 1]^{1/2} x_3) + E \exp(i k [c^2 / \beta^2 - 1]^{1/2} x_3) \qquad (5)$$

and similarly for the other functions of (4).

g, and f and h, will be confined to near the boundary if the exponential arguments are real and negative. So we require $[c^2 / \beta^2 - 1]^{1/2}$ (and similar terms) to be positive imaginary i.e.

$c^2 / \beta^2 < 1$; or $c^2 < \beta^2$; and similarly:

$c^2 < \alpha^2$; $c^2 < \beta'^2$; $c^2 < \alpha'^2$.

We also require E = 0 in M ($x_3 < 0$), as otherwise this term would increase away from the boundary; and similarly $B' = 0$ in M′ ($x_3 > 0$. So the solutions are of the form:

$$\phi = A \exp(- i k [c^2 / \alpha^2 - 1]^{1/2} x_3) \exp[i k (x_1 - c t)]$$

$$\psi = B \exp(- i k [c^2 / \beta^2 - 1]^{1/2} x_3) \exp[i k (x_1 - c t)]$$

$$u_2 = C \exp(- i k [c^2 / \beta^2 - 1]^{1/2} x_3) \exp[i k (x_1 - c t)]$$

or

$$\phi = A \exp(i k [- (c^2 / \alpha^2 - 1)^{1/2} x_3 + x_1 - c t])$$

$$\psi = B \exp(i k [- (c^2 / \beta^2 - 1)^{1/2} x_3 + x_1 - c t])$$

$$u_2 = C \exp(i k [- (c^2 / \beta^2 - 1)^{1/2} x_3 + x_1 - c t])$$

for some constants A, B, C; similarly for medium M ′ :

$$\phi' = D' \exp(i k [(c^2 / \alpha'^2 - 1)^{1/2} x_3 + x_1 - c t])$$

$$\psi' = E' \exp(i k [(c^2 / \beta'^2 - 1)^{1/2} x_3 + x_1 - c t])$$

$$u_2' = F' \exp(i k [(c^2 / \beta'^2 - 1)^{1/2} x_3 + x_1 - c t])$$

Boundary conditions

The displacements and stresses across the interface must match. So we have:

$$u_1 = u_1'$$

$u_2 = u_2'$ which implies $C = F'$.

$$u_3 = u_3'$$

$$S_{33} = S_{33}'$$

$$S_{32} = S_{32}'$$

$$S_{31} = S_{31}'$$

We get the displacements using eqn (1),

$$u_1 = \partial \phi / \partial x_1 - \partial \psi / \partial x_3$$

$$u_3 = \partial \phi / \partial x_3 + \partial \psi / \partial x_1$$

The stresses are given by:

$$S_{33} = 2 \mu \partial u_3 / \partial x_3 + \lambda (\partial u_1 / \partial x_1 + \partial u_3 / \partial x_3)$$

$$S_{31} = \mu (\partial u_3 / \partial x_1 + \partial u_1 / \partial x_3)$$

$$S_{32} = \mu \partial u_2 / \partial x_3$$

- since there is to be no x_2 dependence.

Thus we have:

$$u_1 = i k A \exp(i k [- (c^2 / \alpha^2 - 1)^{1/2} x_3 + x_1 - c t])$$

$$+ i k B (c^2 / \beta^2 - 1)^{1/2} \exp(i k [- (c^2 / \beta^2 - 1)^{1/2} x_3 + x_1 - c t])$$

$$u_3 = - i k A(c^2 / \alpha^2 - 1)^{1/2} \exp(i k [- (c^2 / \alpha^2 - 1)^{1/2} x_3 + x_1 - c t])$$

$$+ i k B \exp(i k [- (c^2 / \beta^2 - 1)^{1/2} x_3 + x_1 - c t])$$

At $x_3 = 0$:

$$\partial u_1 / \partial x_1 = - k^2 \exp(i k [x_1 - c t])\{ A + B(c^2 / \beta^2 - 1)^{1/2}\}$$

$$\partial u_1 / \partial x_3 = - k^2 \exp(i k [x_1 - c t])\{- A (c^2 / \beta^2 - 1)^{1/2} - B(c^2 / \beta^2 - 1) \}$$

$$\partial u_3 / \partial x_1 = - k^2 \exp(i k [x_1 - c t]) \{ - A(c^2 / \alpha^2 - 1)^{1/2} + B \}$$

$$\partial u_3 / \partial x_3 = - k^2 \exp(i k [x_1 - c t])\{A(c^2 / \alpha^2 - 1) - B(c^2 / \beta^2 - 1)^{1/2}\}$$

$$\partial u_2 / \partial x_3 = i k \exp(i k [x_1 - c t]) \{ -C (c^2 / \beta^2 - 1)^{1/2} \}$$

The stress terms are therefore:

$$S_{33} = 2 \mu \, \partial u_3 / \partial x_3 + \lambda (\partial u_1 / \partial x_1 + \partial u_3 / \partial x_3)$$

$$= - 2 \mu k^2 \exp(i k [x_1 - c t])\{A(c^2 / \alpha^2 - 1) - B(c^2 / \beta^2 - 1)^{1/2}\}$$

$$- \lambda k^2 \exp(i k [x_1 - c t])\{ A + B(c^2 / \beta^2 - 1)^{1/2}\}$$

$$- \lambda k^2 \exp(i k [x_1 - c t])\{A(c^2 / \alpha^2 - 1) - B(c^2 / \beta^2 - 1)^{1/2}\}$$

$$= - k^2 \exp(i k[x_1 - c t])[\{2 \mu + \lambda\}\{A(c^2 / \alpha^2 - 1) - B(c^2 / \beta^2 - 1)^{1/2}\} + \lambda \{A+B(c^2 / \beta^2 - 1)^{1/2}\}]$$

$$S_{31} = \mu (\partial u_3 / \partial x_1 + \partial u_1 / \partial x_3)$$

$$= \mu \{ - k^2 \exp(i k [x_1 - c t]) \{ - A(c^2 / \alpha^2 - 1)^{1/2} + B \}\}$$

$$- k^2 \exp(i k [x_1 - c t])\{- A (c^2 / \beta^2 - 1)^{1/2} - B(c^2 / \beta^2 - 1) \} \}$$

$$= \mu k^2 \exp(i k[x_1 - c t])\{2 A(c^2 / \alpha^2 - 1)^{1/2} + B(c^2 / \beta^2 - 2)\}$$

$$S_{32} = \mu \, \partial u_2 / \partial x_3 = i \mu k \exp(i k [x_1 - c t]) \{ - C (c^2 / \beta^2 - 1)^{1/2} \}$$

The corresponding terms in the ′ medium are as follows.

$$u_1' = i k D' \exp(i k [(c^2 / \alpha'^2 - 1)^{1/2} x_3 + x_1 - c t])$$

$$- i k E' (c^2 / \beta'^2 - 1)^{1/2} \exp(i k [(c^2 / \beta'^2 - 1)^{1/2} x_3 + x_1 - c t])$$

$$u_3' = i k D' (c^2 / \alpha'^2 - 1)^{1/2} \exp(i k [(c^2 / \alpha'^2 - 1)^{1/2} x_3 + x_1 - c t])$$

$$+ i k E' \exp(i k [(c^2 / \beta'^2 - 1)^{1/2} x_3 + x_1 - c t])$$

$$S_{33}' = - k^2 \exp(i k[x_1 - c t])[\{2 \mu' + \lambda'\}\{D' (c^2 / \alpha'^2 - 1) + E' (c^2 / \beta'^2 - 1)^{1/2}\}$$

$$+ \lambda' \{D' - E' (c^2 / \beta'^2 - 1)^{1/2}\}]$$

$S_{31}' \quad = - \mu' k^2 \exp(i k[x_1 - c t])\{2D' (c^2 / \alpha'^2 - 1)^{1/2} - E' (c^2 / \beta'^2 - 1) + E' \}$

$S_{32}' \quad = - i \mu' k \exp(i k [x_1 - c t]) \{ - F' (c^2 / \beta'^2 - 1)^{1/2} \}$

Equating terms at $x_3 = 0$, and suppressing terms like
$i k \exp(i k [- (c^2 / \alpha^2 - 1)^{1/2} x_3 + x_1 - c t])$, gives, for displacements

$$\{ A + B(c^2 / \beta^2 - 1)^{1/2}\} = \{ D' - E' (c^2 / \beta'^2 - 1)^{1/2}\}$$

$$\{ - A(c^2 / \alpha^2 - 1)^{1/2} + B \} = \{ D' (c^2 / \alpha'^2 - 1)^{1/2} + E'\} \qquad \text{….(6.1, 6.2)}$$

And for the stresses:

$$\{2 \mu + \lambda\}\{A(c^2 / \alpha^2 - 1) - B(c^2 / \beta^2 - 1)^{1/2}\} + \lambda \{A+B(c^2 / \beta^2 - 1)^{1/2}\}$$

$$= \{2 \mu' + \lambda'\}\{D' (c^2 / \alpha'^2 - 1) + E' (c^2 / \beta'^2 - 1)^{1/2}\} + \lambda' \{D' - E' (c^2 / \beta'^2 - 1)^{1/2}\}$$

$$\mu \{2 A(c^2 / \alpha^2 - 1)^{1/2} + B(c^2 / \beta^2 - 2)\} = - \mu'\{2D' (c^2 / \alpha'^2 - 1)^{1/2} - E' (c^2 / \beta'^2 - 2) \}$$

$$- \mu C (c^2 / \beta^2 - 1)^{1/2} = \mu' F' (c^2 / \beta'^2 - 1)^{1/2} \qquad \text{…(6.3, 6.4, 6.5)}$$

Since $C = F'$, this last equation 6.5 implies $C = F' = 0$, because of the sign difference. This gives a most important result: that there are no waves of the kind we are seeking with a u_2 component when there is a single interface. NB surface waves of this kind do exist when there are multiple layers. They are called Love Waves.

The balance of the 4 equations enables us to solve for the relationship between the unknowns A, B, D′ ,E′ and c. Note that we can eliminate the unknowns μ and λ using

$$\alpha^2 = (2 \mu + \lambda) /\rho,$$

$$\beta^2 = \mu /\rho, \quad \text{(whence } \lambda = \rho(\alpha^2 - 2\beta^2) \text{) where } \rho \text{ is the density.}$$

Even though the equations can be simplified (a bit), the algebra is *gruesome.*

Rayleigh waves

Instead of considering the gruesome general case further, we look at the special case where the boundary is a free surface e.g. the Earth's surface. Then the material properties in medium M′ all have zero values. In particular, the stresses on the free side are all zero (what could cause them?) and equations 6.3, 6.4 become:

$$\{2 \mu + \lambda\}\{A(c^2 / \alpha^2 - 1) - B(c^2 / \beta^2 - 1)^{1/2}\} + \lambda \{A+B(c^2 / \beta^2 - 1)^{1/2}\} = 0$$

$$\mu \{2 A(c^2 / \alpha^2 - 1)^{1/2} + B(c^2 / \beta^2 - 2)\} = 0$$

Simplifying, and eliminating μ and λ :

$$A (c^2 - 2\beta^2) - B (2\beta^2) (c^2 / \beta^2 - 1)^{1/2}) = 0$$

and

$$2 A(c^2 / \alpha^2 - 1)^{1/2} + B(c^2 / \beta^2 - 2) = 0$$

Eliminating A and B gives:

$$(A/B)^2 = 4\beta^4(c^2/\beta^2 - 1) / (c^2 - 2\beta^2)^2 = (c^2/\beta^2 - 2)^2 / 4(c^2/\alpha^2 - 1) \qquad (7)$$

or

$$16(c^2/\beta^2 - 1)(c^2/\beta^2 - \alpha^2/\beta^2)(\beta^2/\alpha^2) - (c^2/\beta^2 - 2)^4 = 0$$

simplifying:

$$(c^2/\beta^2)^3 - 8(c^2/\beta^2)^2 + (24 - 16\beta^2/\alpha^2)(c^2/\beta^2) - 16(1 - \beta^2/\alpha^2) = 0 \qquad (8)$$

which is a cubic in (c^2/β^2) and therefore has at least 1 real root. Putting $c = 0$ and $c = \beta$ into the LHS of 8 gives $-16(1 - \beta^2/\alpha^2) < 0$ and $1 - 8 + 24 - 16 = 1 > 0$. So there is a root between $c = 0$ and $c = \beta$. This satisfies the requirement stated earlier that $c/\beta < 1$. For normal values of β/α, we get a root $c \sim 0.9\beta$.

Motion of a Rayleigh wave on the Earth's surface

If we go back to the displacements:

$$u_1 = ikA\exp(ik[-(c^2/\alpha^2 - 1)^{1/2} x_3 + x_1 - ct])$$

$$+ ikB(c^2/\beta^2 - 1)^{1/2}\exp(ik[-(c^2/\beta^2 - 1)^{1/2} x_3 + x_1 - ct])$$

$$u_3 = -ikA(c^2/\alpha^2 - 1)^{1/2}\exp(ik[-(c^2/\alpha^2 - 1)^{1/2} x_3 + x_1 - ct])$$

$$+ ikB\exp(ik[-(c^2/\beta^2 - 1)^{1/2} x_3 + x_1 - ct])$$

Consider the displacement at the surface ($x_3 = 0$) at some fixed $x_1 = 0$, say. Then we have:

$$u_1 = ikA\exp(-ikct) + ikB(c^2/\beta^2 - 1)^{1/2}\exp(-ikct)$$

$$u_3 = -ikA(c^2/\alpha^2 - 1)^{1/2}\exp(-ikct) + ikB\exp(-ikct)$$

Equation 7 gives us:

$$(A/B)^2 = (c^2/\beta^2 - 2)^2 / 4(c^2/\alpha^2 - 1),$$

or

$$(A/B) = \pm(2 - c^2/\beta^2) / 2(c^2/\alpha^2 - 1)^{1/2}, \qquad (9)$$

and we presume we have solved for c/β.

Since $c/\alpha < 1$, $A/B = \pm i(2 - c^2/\beta^2) / 2(1 - c^2/\alpha^2)^{1/2} = i\gamma$, say, or $iA = -(\pm)\gamma B$.

From this, $(1 - c^2/\alpha^2)^{1/2} = (2 - c^2/\beta^2) / 2\gamma$.

So:

$$u_1 = -k\gamma B\exp(-ikct) + ikB(c^2/\beta^2 - 1)^{1/2}\exp(-ikct)$$

$$u_3 = k \gamma B (c^2 / \alpha^2 - 1)^{1/2} \exp(- i k c t) + i k B \exp(- i k c t)$$

Now rewrite $(c^2 / \beta^2 - 1)^{1/2}$ as $i (1 - c^2 / \beta^2)^{1/2}$; similarly for $(c^2 / \alpha^2 - 1)^{1/2}$;
use $(1 - c^2 / \alpha^2)^{1/2} = (2 - c^2 / \beta^2) / 2 \gamma$; write $\omega = - k c$;
and take a unit value of $- kB$ i.e put $- kB = 1$.

$$u_1 = \gamma \exp(i \omega t) + (1- c^2 / \beta^2)^{1/2} \exp(i \omega t)$$

$$u_3 = - i (2 - c^2 / \beta^2) /2 \exp(i \omega t) - i \exp(i \omega t) \quad (\gamma \text{ 's cancelled})$$

or

$$u_1 = U_1 \exp(i \omega t)$$

$$u_3 = i U_3 \exp(i \omega t)$$

where $U_1 = \gamma + (1- c^2 / \beta^2)^{1/2}$; $U_3 = - (2 - c^2 / \beta^2) /2 - 1$; both are real.

Now write $i = \exp(i \pi/2)$, substitute and collect terms:

$$u_1 = U_1 \exp(i \omega t)$$

$$u_3 = U_3 \exp(i (\omega t + \pi/2))$$

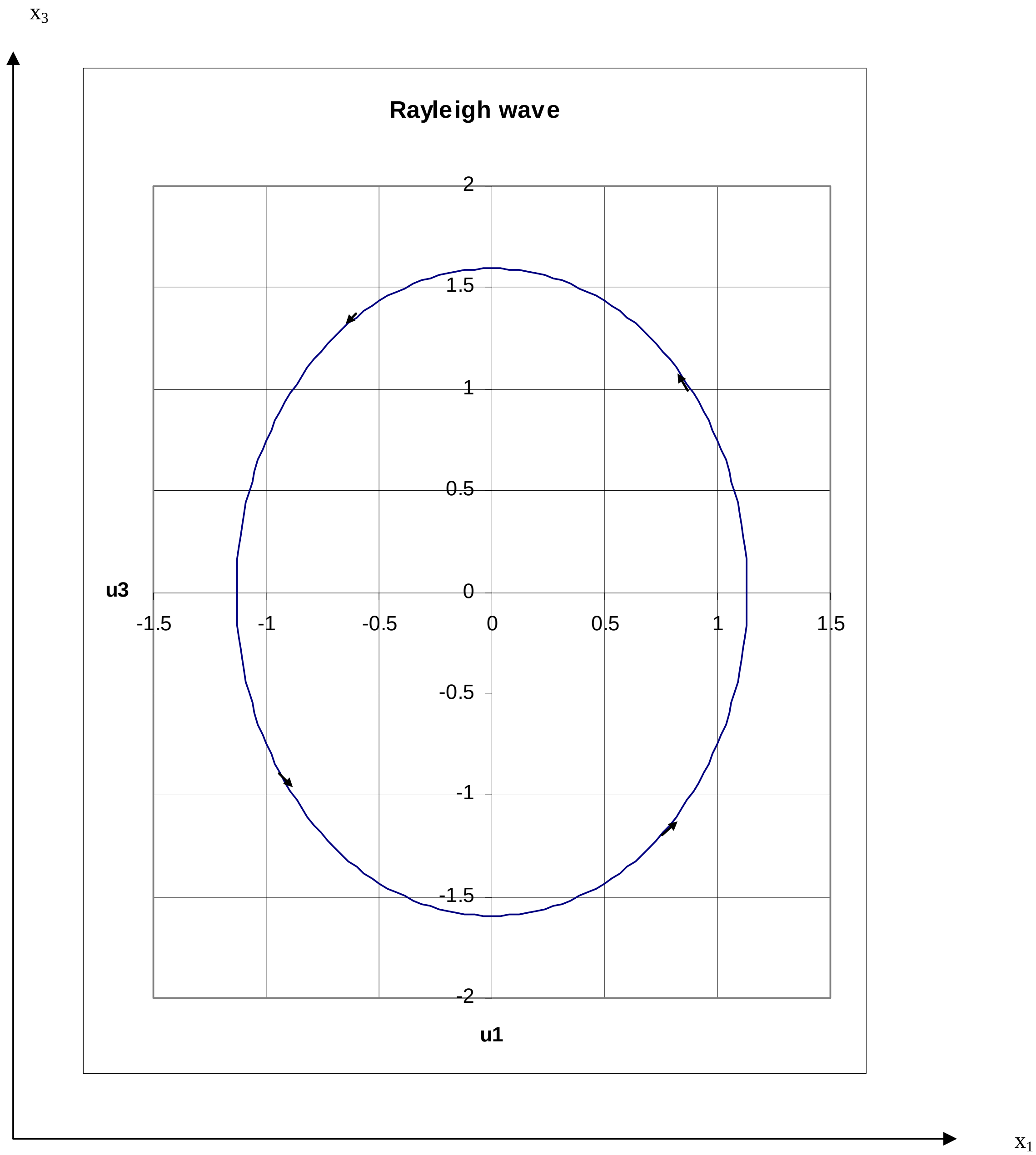
Now $\cos (\omega t + \pi/2) = \cos \omega t \cos \pi/2 - \sin \omega t \sin \pi/2 = - \sin \omega t$

The real parts of the displacement are thus

$$u_1 = U_1 \cos(\omega t)$$

$$u_3 = - |U_3| \sin(\omega t) \qquad \text{(NB as defined above } U_3 \text{ will be negative)}$$

which describes an ellipse as a function of time. The medium is displaced in a retrograde way as shown. For $\beta = 3.4$ km/s, take $\alpha = \sqrt{3} \beta$; $c \sim 3.06$ km/s and hence $U_1 = 1.13$; $U_3 = -1.60$.


x3
Rayleigh wave
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
u3
-1.5
-1
-0.5
0
0.5
1
1.5
u1
x1

Appendix: Transformation to non-Cartesian coordinates

The position vector $\underline{\mathbf{r}}$ = (x, y, z) at point P can be written as a function of any set of coordinates u_j

$$\underline{\mathbf{r}} = \underline{\mathbf{r}}(u_1, u_2, u_3).$$

A tangent vector to the u_1 curve at P (u_2, u_3 = constants) is given by

$\partial \underline{\mathbf{r}} / \partial u_1$, so a unit vector in this direction is

$$\underline{\mathbf{e}}_1 = \partial \underline{\mathbf{r}} / \partial u_1 / | \partial \underline{\mathbf{r}} / \partial u_1 |$$

This is the direction of increasing u_1 . Similarly for $\underline{\mathbf{e}}_2$ and $\underline{\mathbf{e}}_3$. This gives us the direction of the coordinate axes, at $\underline{\mathbf{r}}(u_1, u_2, u_3)$.

Write $h_j = | \partial \underline{\mathbf{r}} / \partial u_j |$; these are called *scale factors.*

∇f

We want to write:

$$\nabla f = \underline{\mathbf{e}}_1 f_1 + \underline{\mathbf{e}}_2 f_2 + \underline{\mathbf{e}}_3 f_3$$

where the f_j are to be determined. We have:

$$d\underline{\mathbf{r}} = \partial \underline{\mathbf{r}} / \partial u_1 \, du_1 + \partial \underline{\mathbf{r}} / \partial u_2 \, du_2 + \partial \underline{\mathbf{r}} / \partial u_3 \, du_3$$

$$= \underline{\mathbf{e}}_1 | \partial \underline{\mathbf{r}} / \partial u_1 | du_1 + \underline{\mathbf{e}}_2 | \partial \underline{\mathbf{r}} / \partial u_2 | du_2 + \underline{\mathbf{e}}_3 | \partial \underline{\mathbf{r}} / \partial u_3 | du_3$$

$$= \underline{\mathbf{e}}_1 h_1 \, du_1 + \underline{\mathbf{e}}_2 h_2 \, du_2 + \underline{\mathbf{e}}_3 h_3 \, du_3$$

Write df two ways:

$$df = \partial f / \partial x \, dx + \partial f / \partial y \, dy + \partial f / \partial z \, dz$$

$$= \nabla f \bullet d\underline{\mathbf{r}}$$

$$= (\underline{\mathbf{e}}_1 f_1 + \underline{\mathbf{e}}_2 f_2 + \underline{\mathbf{e}}_3 f_3) \bullet (\underline{\mathbf{e}}_1 h_1 \, du_1 + \underline{\mathbf{e}}_2 h_2 \, du_2 + \underline{\mathbf{e}}_3 h_3 \, du_3)$$

$$= f_1 h_1 du_1 + f_2 h_2 du_2 + f_3 h_3 du_3$$

and

$$df = \partial f / \partial u_1 \, du_1 + \partial f / \partial u_2 \, du_2 + \partial f / \partial u_3 \, du_3$$

Hence, comparing the two:

$$\left((1/h_1) \partial f / \partial u_1 , (1/h_2) \partial f / \partial u_2 , (1/h_3) \partial f / \partial u_3 \right) = (f_1, f_2, f_3)$$

So the LHS is ∇f in the new coordinate system, and the operator ∇ is given by:

$$\nabla = \left((1/h_1) \partial / \partial u_1 , (1/h_2) \partial / \partial u_2 , (1/h_3) \partial / \partial u_3 \right)$$

Spherical Polar Coordinates

In spherical polar coordinates:

$$(x, y, z) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$$

$$(u_1, u_2, u_3) = (r, \theta, \phi)$$

$$h_1 = |\partial \underline{\mathbf{r}} / \partial r| = 1$$

$$h_2 = |\partial \underline{\mathbf{r}} / \partial \theta| = r(\cos^2\theta \cos^2\phi + \cos^2\theta \sin^2\phi + \sin^2\theta)^{1/2} = r$$

$$h_3 = |\partial \underline{\mathbf{r}} / \partial \phi| = r(\sin^2\theta \sin^2\phi + \sin^2\theta \cos^2\phi + 0)^{1/2} = r \sin\theta$$

So:

$$\nabla f = (\partial f / \partial r, (1/r) \partial f / \partial \theta, (1/r \sin\theta) \partial f / \partial \phi)$$

$\nabla \bullet \underline{\mathbf{v}}$

It can be shown (tutorial exercise) that

$$\nabla \bullet \underline{\mathbf{v}} = (1/h_1 h_2 h_3) \{ \partial (h_2 h_3 v_1)/\partial u_1 + \partial (h_3 h_1 v_2)/\partial u_2$$

$$+ \partial (h_1 h_2 v_3)/\partial u_3 \}$$

In spherical polars:

$$\nabla \bullet \underline{\mathbf{v}} = (1/r^2 \sin\theta) \{ \partial (r^2 \sin\theta\, v_r)/\partial r + \partial (r \sin\theta\, v_\theta)/\partial \theta$$

$$+ \partial (r v_\phi)/\partial \phi \}$$

$\nabla^2 f$ in spherical polars

$$\nabla \bullet \nabla f = (1/r^2)\, \partial/\partial r\, (r^2 \partial f/\partial r) + (1/r^2 \sin\theta)\, \partial/\partial\theta\, (\sin\theta\, \partial f/\partial\theta)$$

$$+ (1/r^2 \sin^2\theta)\, \partial^2 f/\partial^2 \phi$$