GPHS/MATH 323 Mathematics for Earth Science PDEs III

The second important PDE we will consider is Laplace's equation:

$$\nabla^2 u = 0 \equiv \partial^2 u / \partial x_j \, \partial x_j = 0$$

where u can be a scalar or a vector.

Laplace's equation arises in many circumstances. In particular, following Morse and Fessback, it arises when there is a steady state reached in a problem
(1)

$$\partial^2 u / \partial t^2 = c^2 \, \partial^2 u / \partial x_j \, \partial x_j \qquad \text{(wave equation)} \qquad (1)$$

or

$$\partial u / \partial t = k \, \partial^2 u / \partial x_j \, \partial x_j \qquad \text{(diffusion equation)} \qquad (2)$$

In the steady state, or if we can neglect inertia effects in (1) because the accelerations are so small, we have

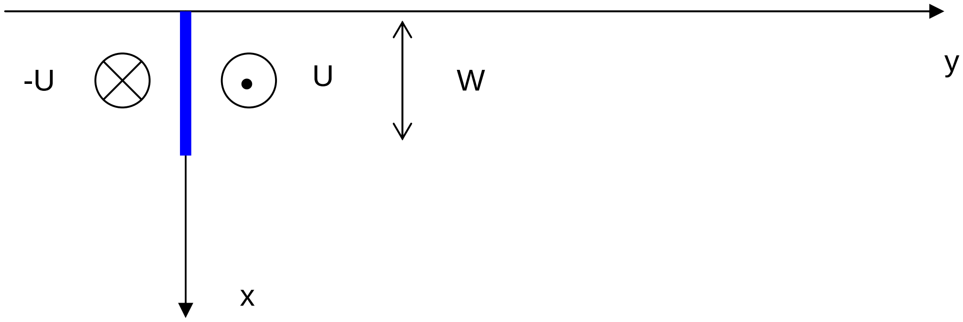
$$0 = c^2 \, \partial^2 u / \partial x_j \, \partial x_j \qquad \text{(Laplace's equation)} \qquad (1)$$

Our example for Laplace's equation is taken from one of these circumstances.

We are interested in how the Earth deforms before and during an earthquake. We have already considered the waves that are generated by earthquakes, but after the waves have propagated away from the source of the earthquake (fault) what deformation is left? And what deformation might there have been before the earthquake that would have led it the fault's failure?

Earthquake faulting

Imagine we have a very long (infinite), vertical fault in a semi-infinite, isotropic half-space, which is displaced in the direction of its strike (z axis; - a 'strike slip' fault). Note that we will be using (x, y, z) interchangeably with (x_1, x_2, x_3).



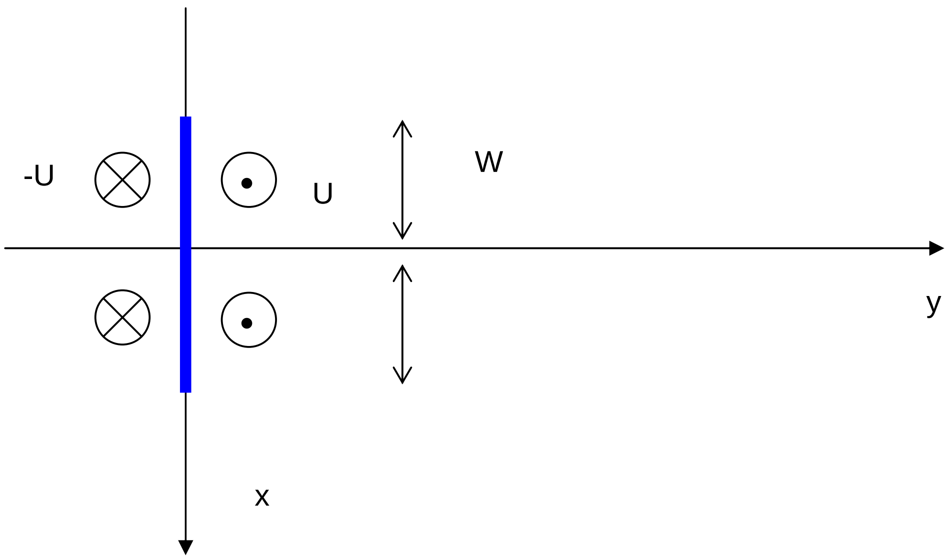
Fiig 1 Why we should be interested…The Wellington fault.

The displacement is U in the +z direction at $y=0^+$ and U in the –z direction for $y=0^-$ over the range $0 \le x \le W$. U << W e.g. U o(m), W o(10km).

We assume that the displacement is that same at all z i.e there is no change in the z direction (i.e. $\partial(\ )/\partial z = 0$.)

Problem: what are the displacements $u_z(x,y)$ within the Earth; especially, what are the displacements at x = 0?

We shall solve this by the *method of images*. We shall create an image of the fault by reflecting it in the yz plane:



We will verify later that this does not 'harm' the problem i.e *it does not change the boundary conditions.*

Boundary conditions

These are: (1) $u_z(x,0^+) = U$ on $0 \le x \le W$, and (2) the stresses across the yz plane must be zero i.e. $S_{xj} = 0$ for j = x, y, z.

We observe that the problem is (anti-)symmetric: the space y < 0 is the negative of the space $y \ge 0$. So we shall solve the problem in the half space $y \ge 0$.

The displacements must have obeyed Navier's equation:

$$\rho\, \partial^2 u_z / \partial t^2 = \mu\, \partial^2 u_z / \partial x_j \partial x_j + (\mu + \lambda)\, \partial^2 u_j / \partial x_j \partial x_z$$

We assume that (i) the rock is incompressible (reasonable on a time scale longer than the passage of waves) and that (ii) the Earth is now in equilibrium

i.e. $\rho\, \partial^2 u_z / \partial t^2 = 0$

Incompressibility implies that the dilatation, $\partial u_z / \partial x_z$, is zero. So:

$$0 = \mu \partial^2 u_z / \partial x_j \partial x_j$$

which is Laplace's equation for u_z .

Write this out:

$$\partial^2 u_z / \partial x^2 + \partial^2 u_z / \partial y^2 = 0 \quad \text{(NB no change in the z direction)}$$

We will initially solve this for a more general set of boundary conditions:

(1) $u_z(x,0) = f(x)$, f(x) piecewise smooth (a finite number of steps is OK, so our problem, qualifies)

(2) $u_z(x,y)$ is *bounded* in $y \geq 0$. This is reasonable for the real world!

We will solve the problem by *Fourier transforms* (solution from "Partial Differential Equations and boundary value problems with applications" by Mark A Pinsky, McGraw-Hill QA 374 P658 p 3ed).

Recall that for 'nice' functions like f(x) we can write

$$F(\nu) = \int_{-\infty}^{\infty} f(x) \exp(- i \nu x)\, dx$$

and that we can recover the original function with the inverse transform:

$$f(x) = (1/2\pi) \int_{-\infty}^{\infty} F(\nu) \exp(i \nu x)\, d\nu$$

we use 'ν ' instead of ω because to emphasise that this is a *spatial* transform..

Now we look for a solution in the form:

$$u_z(x,y) = (1/2\pi) \int_{-\infty}^{\infty} F(\nu, y) \exp(i \nu x)\, d\nu$$

Apply the Laplacian operator:

$$\partial^2 u_z / \partial x^2 + \partial^2 u_z / \partial y^2 =$$

$$(1/2\pi) \int_{-\infty}^{\infty} - \nu^2 F(\nu, y) \exp(i \nu x)\, d\nu + (1/2\pi) \int_{-\infty}^{\infty} \partial^2 F(\nu, y)/ \partial y^2 \exp(i \nu x)\, d\nu$$

$$=(1/2\pi) \int_{-\infty}^{\infty} [\partial^2 F(\nu, y)/ \partial y^2 - \nu^2 F(\nu, y)] \exp(i \nu x)\, d\nu$$

which = 0 iff the expression in [ ] is zero, i.e. we have the ODE

$$\partial^2 F(\nu, y)/\partial y^2 - \nu^2 F(\nu, y) \quad = 0.$$

This has a general solution:

$$F(\nu, y) = A \exp(\nu y) + B \exp(-\nu y)$$

Applying the boundedness condition means A = 0 for $\nu > 0$ and B = 0 for $\nu < 0$.

So for $\nu < 0$ at y = 0,

$$F(\nu, 0) = F(\nu) = A \exp(0), \text{ so } A = F(\nu).$$

Similarly, for $\nu > 0$, B = $F(\nu)$.

So we can write:

$$F(\nu, y) = F(\nu).\exp(-|\nu| y), \qquad \text{where } F(\nu) \text{ means } F(\nu, 0).$$

So,

$$u_z(x,y) = (1/2\pi) \int_{-\infty}^{\infty} F(\nu).\exp(-|\nu| y) \exp(i \nu x)\, d\nu$$

Now, also,

$$F(\nu) = \int_{-\infty}^{\infty} f(x) \exp(-i \nu x)\, dx \qquad \text{because } u_z(x,0) = f(x),$$

Substitute for $F(\nu)$:

$$u_z(x,y) = (1/2\pi) \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\xi) \exp(-i \nu \xi)\, d\xi \right] \exp(-|\nu| y) \exp(i \nu x)\, d\nu$$

where we have had to change the integration dummy variable in the [ ] integral.

Isn't this magic: we have a solution! F is gone! And we only need the values of f( ) on the x axis, which we have (boundary condition).

Rearrange the order of the two integrals:

$$u_z(x,y) = (1/2\pi) \int_{-\infty}^{\infty} f(\xi) \left[\int_{-\infty}^{\infty} \exp(-i \nu \xi) \exp(-|\nu| y) \exp(i \nu x)\, d\nu \right] d\xi$$

Now consider the inner one:

$$I(x,y,\xi) = \int_{-\infty}^{\infty} \exp(-i \nu \xi) \exp(-|\nu| y) \exp(i \nu x)\, d\nu$$

Collect terms

$$I(x,y,\xi) = \int_{-\infty}^{\infty} \exp(i \nu[x-\xi]) \exp(-|\nu| y)\, d\nu$$

…and now we will get rid of $|\nu|$:

$$I(x,y,\xi) = \int_{-\infty}^{0} \exp(i\nu[x-\xi]) \exp(-|\nu| y)\, d\nu + \int_{0}^{\infty} \exp(i\nu[x-\xi]) \exp(-|\nu| y)\, d\nu$$

In the first integral, substitute $-\zeta = \nu$. $d\zeta = -d\nu$; and change the order of integration limits; this takes out the minus sign.

$$I(x,y,\xi) = \int_{0}^{\infty} \exp(-i\zeta[x-\xi]) \exp(-|\zeta| y)\, d\zeta + \int_{0}^{\infty} \exp(i\nu[x-\xi]) \exp(-|\nu| y)\, d\nu$$

Resubstitute $\nu = \zeta$.

$$I(x,y,\xi) = \int_{0}^{\infty} \exp(-i\nu[x-\xi]) \exp(-|\nu| y)\, d\nu + \int_{0}^{\infty} \exp(i\nu[x-\xi]) \exp(-|\nu| y)\, d\nu$$

Expand the complex exponentials as cos() + i sin()

$$I(x,y,\xi) = \int_{0}^{\infty} [\cos(-\nu[x-\xi]) + i\sin(-\nu[x-\xi])] \exp(-|\nu| y)\, d\nu$$

$$+ \int_{0}^{\infty} [\cos(\nu[x-\xi]) + i\sin(\nu[x-\xi])] \exp(-|\nu| y)\, d\nu$$

Now the sin terms have opposite signs, so their contribution to the integral is zero. We are left with the 2 (real) cos terms; so write:

$$I(x,y,\xi) = 2 R_e \int_{0}^{\infty} \exp(i\nu[x-\xi]) \exp(-|\nu| y)\, d\nu$$

or

$$I(x,y,\xi) = 2 R_e \int_{0}^{\infty} \exp(-\nu\{y - i[x-\xi]\})\, d\nu$$

where we can drop the modulus since the integration range is now positive.
This can be integrated:

$$I(x,y,\xi) = 2 R_e \left[\frac{-\exp(-\nu\{y - i[x-\xi]\})}{\{y - i[x-\xi]\}} \right]_{0}^{\infty}$$

$$= 2 R_e \frac{1}{(y - i[x-\xi])} \cdot \frac{(y + i[x-\xi])}{(y + i[x-\xi])}$$

(rationalising)

$$= \quad 2y / (y^2 + [x-\xi]^2)$$

Substitute back in the integral for $u_z(x,y)$:

$$u_z(x,y) = (1/2\pi) \int_{-\infty}^{\infty} 2\, y / (y^2 + [x- \xi]^2)\, f(\xi)\, d\xi \text{ (and cancel the '2's)} \qquad \text{(GS)}$$

So this is the *general solution to the problem with general (bounded) f(x).*

Now return to our specific problem, with f(x) given by:

$$u_z(x,0) = f(x) = U \text{ on } 0 \le x \le W, 0 \text{ elsewhere.}$$

So:

$$u_z(x,y) = (U/\pi) \int_{-W}^{W} y / (y^2 + [x- \xi]^2)\, d\xi \qquad \text{(U is indept.of } \xi)$$

Change variable : write $q = (\xi - x) /y$; $dq = (1/y)\, d\xi$; NB $y \neq 0$;

(In the integral $x- \xi$ is squared; so reversing the order is OK)

Then, after dividing top and bottom by y^2:

$$u_z(x,y) = (U/\pi) \int_{(-W - x)/y}^{(W - x)/y} 1 / (1 + q^2)\, dq$$

Now recall $\int 1 / (1 + q^2)\, dq = \arctan (q)$

[For proof: write $x = \tan y = \sin y/\cos y$, so $dx/dy = 1 + \tan^2 y$;
so $dy/dx = d \arctan(x)/dx = 1/(1 + \tan^2 y) = 1/(1 + x^2)$.]

So:

$$u_z(x,y) = (U/\pi) [\arctan(q)]_{(-W - x)/y}^{(W - x)/y}$$

$$= (U/\pi) [\arctan((W - x)/y) - \arctan((-W - x)/y)] ; \quad y \neq 0$$

$$= (U/\pi) [\arctan((W - x)/y) + \arctan((W + x)/y)] ; \quad y \neq 0$$

This is the solution to our problem. Application of the result requires care with the potential singularity at y = 0. However, arctan (1/y) has a limit as $y \to 0^+$:

$$\arctan (1/y) \to \pi/2 \text{ as } y \to 0^+.$$

Testing the model: data from earthquakes

We can (usually) only measure the deformation of the Earth at its surface i.e. at x = 0. On x = 0,

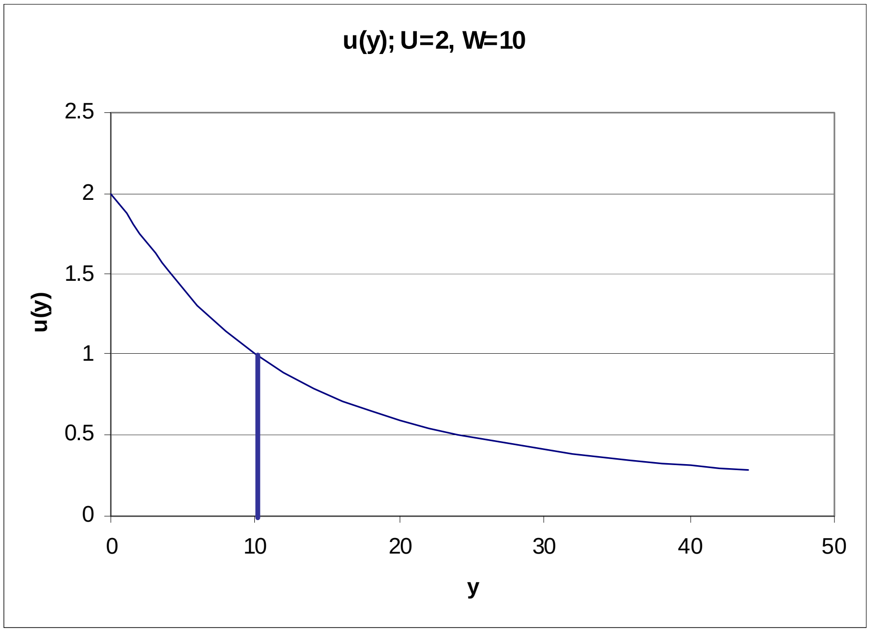
$$u_z(0,y) = (U/\pi) [\arctan(W/y) - \arctan(-W/y)] ; \quad y \neq 0$$

$= U (1/(\pi/2)) \arctan(W/y) \quad y \neq 0$;

As noted above, arctan has a limit of $\pi/2$ as $y \rightarrow 0^+$, so

$u_z(0,y) \rightarrow U$ as $y \rightarrow 0^+$ as it should.

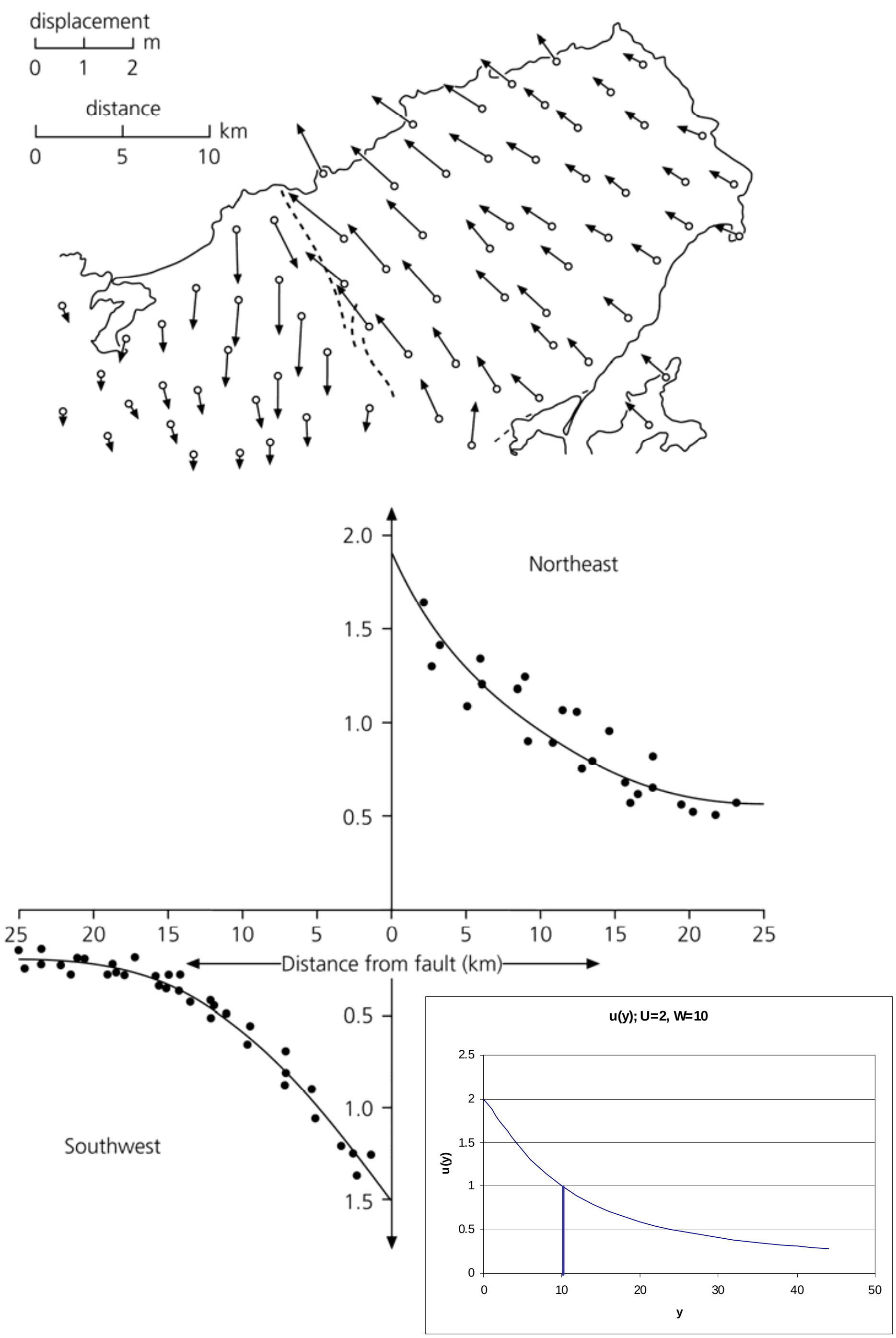
We can plot $u_z(0,y)$:



Note that U(y) = ½ U at y = W.

We can compare this model with observations. (courtesy Michael Wysession, 2003)

Figure 4.5-4: Static displacements for the 1927 Tango, Japan, earthquake.



Model from above

Interpretation

The 1927 tango earthquake in Japan occurred on a (long) strike slip fault that extended about 10km into the Earth.

Checking the validity of the 'images' method

We need to check that the two half planes x > 0 and x < 0 do not interfere with each other. This will be so if the displacements match across x=0 (they do; $u_z(x,y)$ is continuous in x) and that the stresses are zero on x = 0.

That is, we want $S_{xj} = 0$ for j = x, y, z.

From Hooke's law, since the dilatation $E_{jj} = 0$ by assumption,

$$S_{xj} = 2 \mu E_{xj} = \mu (\partial u_x / \partial x_j + \partial u_j / \partial x_x)$$

The only non-zero displacement is u_z so the only possible non-zero stress component is

$$S_{xz} = \mu \partial u_z / \partial x_x$$

$$\partial u_z / \partial x_x \quad = (U/\pi) (1/ (1 + [(W - x)/y]^2) - 1/ (1 + [(-W - x)/y]^2)$$

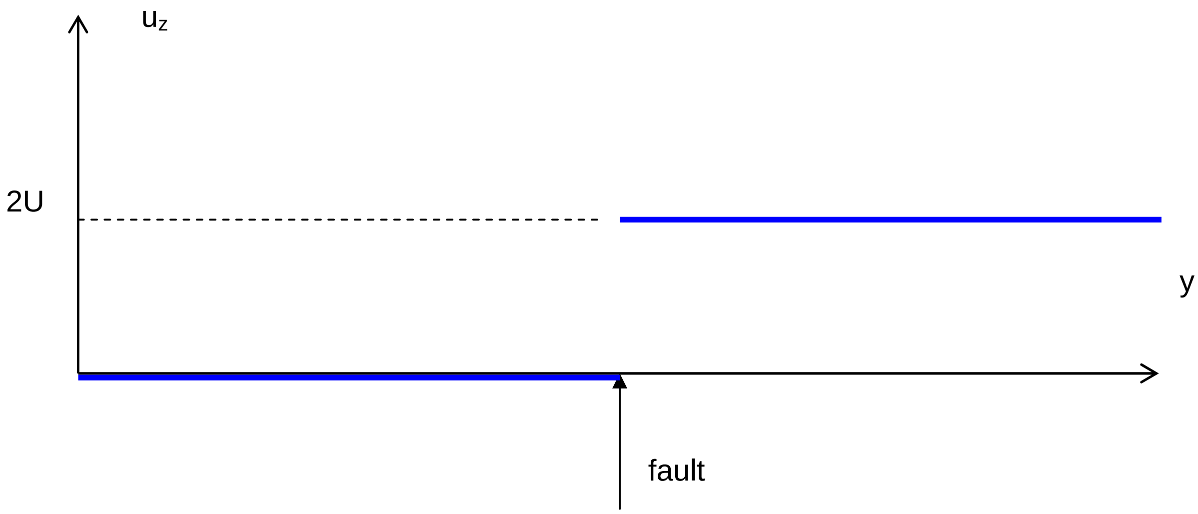
$$= 0 \text{ on } x = 0. \quad \text{QED}$$

Another application of the model; another check with data

After the 1906 San Francisco earthquake, Reid postulated his famous 'elastic rebound' model for earthquakes, which said that earthquakes relaxed the elastic strain that had accumulated between earthquakes.

According to this model there should be no elastic strain left after an earthquake. This must mean that the displacement everywhere is the same as that at the fault, i.e. u = 0 for y <0, u = 2U for y >0. So we have the picture after the earthquake:

In plane x = 0



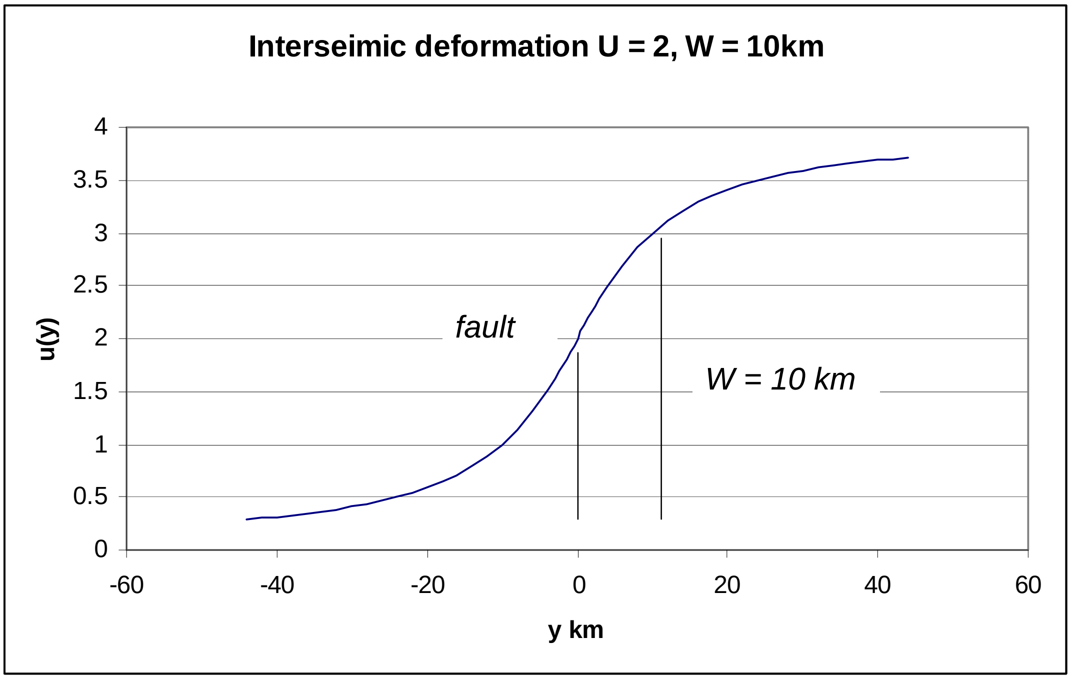
This can be written as a Heaviside function:

$u_z\ (y) = 2U\ H(y)$ (where the Heaviside function H(y) steps by 1 at y = 0).

This means that we can calculate the deformation between earthquakes (*interseismic deformation*) as the difference between the above and the fault displacement (*co-seismic deformation*) which we have just calculated

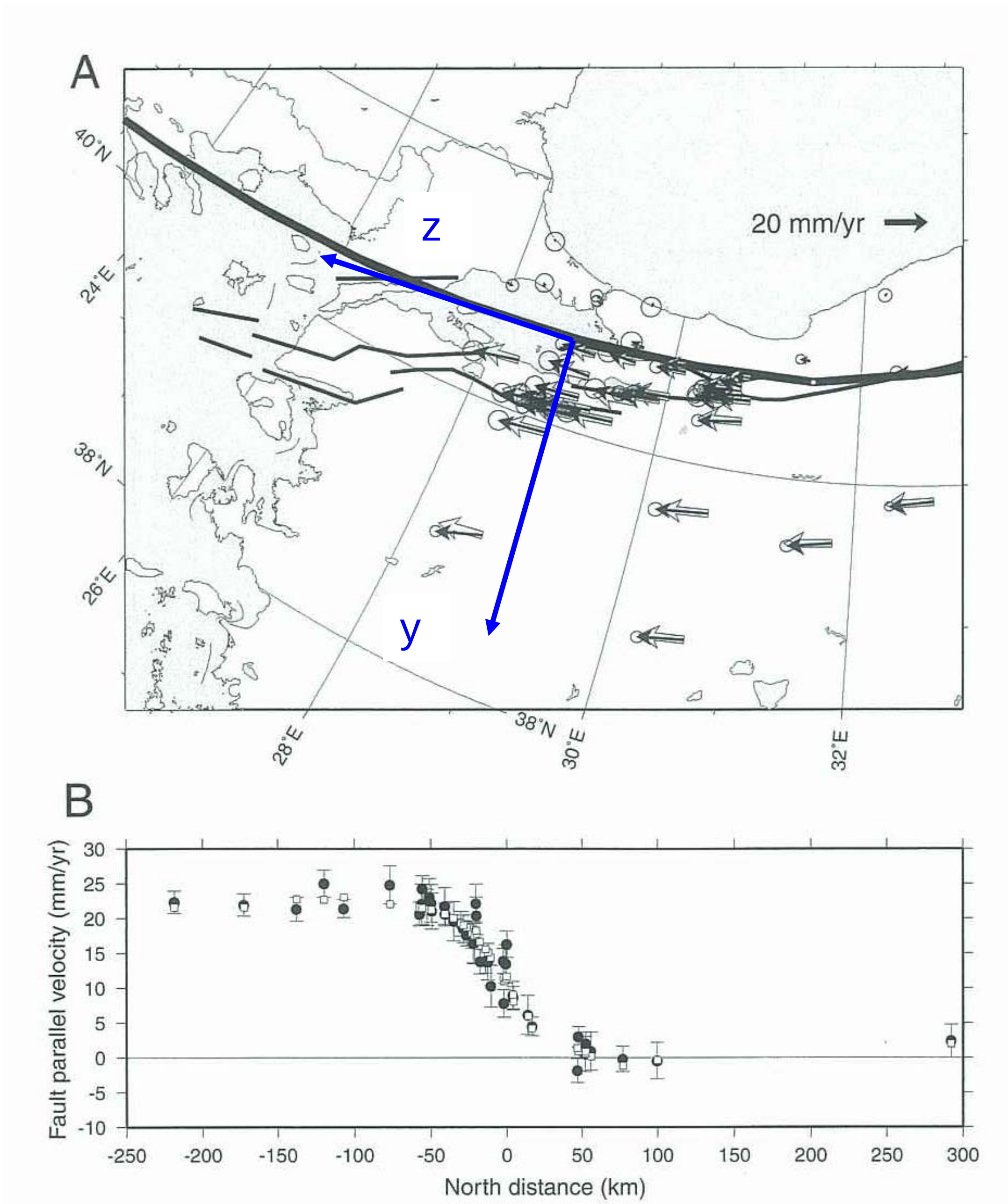
So the displacement of the surface between earthquakes (interseismic deformation) ui_z (y) is:

$$ui_z(y) = 2U\,H(y) - U\,(1/\pi/2)\arctan(W/y)$$



Data : 1999 Izmit, Turkey earthquake (Burgmann et al. 2002)

Interseismic deformation (mm/yr) inferred from GPS data.



In this plot dU/dt is used, rather than U. The rate is assumed constant, so
U = dU/dt x (average number of years between earthquakes)

The figure shows that the interseismic displacement becomes small at about = 50 km from the fault. Comparison with the model suggests 5W ~ 50 km, so W ~ 10 km, similar to Tango 1927.