GPHS/MATH 323 Mathematics for Earth Science VI

Laplace transforms

Our final topic will the Laplace transforms and their role in solving differential equations of certain types.

Text used: H F Weinberger “A first course in partial differential equations” Blaisdell Publishing, 1965 (!)

Definition of the Laplace transform, and its properties

Laplace transforms have some similarities with Fourier transforms. They are defined thus:

Let f(x) be a function *which is zero for x < 0*. The Laplace transform of a function f , $\mathcal{L}$[f](s), is defined by

$$\mathcal{L}[\,f\,](s) = \int_0^\infty \exp(-sx)\, f(x)\, dx \qquad 1$$

So *for such a function*, $\mathcal{L}$[f](s) = $\mathcal{F}$ [f](is), where i = √(-1) and $\mathcal{F}$ is the Fourier transform.

Integrate equation 1 by parts:

$$\mathcal{L}[\,f\,](s) = \left[-(1/s)\exp(-sx)\, f(x)\right]_0^\infty + (1/s)\int_0^\infty \exp(-sx)\, f'(x)\, dx$$

$$\mathcal{L}[\,f\,](s) = (1/s)\, f(0) \quad + (1/s)\int_0^\infty \exp(-sx)\, f'(x)\, dx$$

Or $\displaystyle\int_0^\infty \exp(-sx)\, f'(x)\, dx = s\,\mathcal{L}[\,f\,](s) - f(0)$

provided f(x) is 'well behaved' and is bounded as $x \rightarrow \infty$.

Hence

$$\int_0^\infty \exp(-sx)\, f''(x)\, dx = s\,\mathcal{L}[\,f'\,](s) - f'(0)$$

Hence

$$\int_0^\infty \exp(-sx)\, f''(x)\, dx = s\,\mathcal{L}[\,f'\,](s) - f'(0)$$

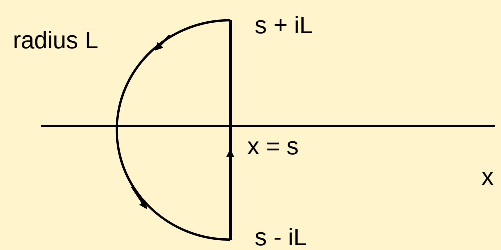
$$= s^2\,\mathcal{L}[\,f\,](s) - s\, f(0) - f'(0)$$

and so on for higher derivatives.

There is an inverse transform, as there is for the Fourier transform.

$$f(x) = (1/2\pi\, i) \lim_{L \rightarrow \infty} \int_{s - iL}^{s + iL} \exp(sx)\,\mathcal{L}[\,f\,](s)\, ds$$

The path of integration is a line in the complex plane at ‘x’ = s. In practice, this integral may be calculated by *completing the contour* with a circle centred at (s, 0) of radius L, when the value of the integral is calculated from the singularities (poles) of L[f](s) in the complex plane using Cauchy’s theorem and the Calculus of Residues. These require more complex variable theory than we have time to do rigorously, but they are very important, powerful methods for calculating integrals. Instead we will obtain inverse transforms from tabulated results! These can all be verified by calculating the transform of the first column function



Some common Laplace transforms

f(x)	$\mathcal{L}$[f](s)
1	1/s
x^n	$n! / s^{n+1}$
exp(ax) ($a \neq 0$)	1/(s-a)
sin ax	$a/(a^2 + s^2)$
cos ax	$s/(a^2 + s^2)$
$x^{1/2}$	½ sqrt(π / s^3)
$x^{-1/2}$	$\sqrt{(\pi / s)}$
x^n exp(-ax)	$n!/(s+a)^{n+1}$, s > -a
f ’ (x)	s $\mathcal{L}$[f](s) - f (0)
$J_0(x)$	$(1 + s^2)^{-1/2}$

Application to PDEs

The ability to write down the Laplace transform of the nth derivative of a function (which is 0 for its argument less than zero) suggests an application: finding the solution to *initial value problems*.

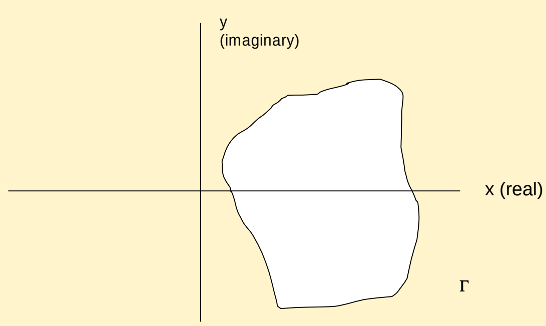
We have stressed that solutions to specific problems will be determined by boundary conditions, and have considered a number of problems with both spatial (boundary interface waves, fault displacement) and spatio-temporal conditions (ocean cooling).

Where the boundary conditions are that the situation is specified at $t = 0$ (or $x_j = 0$) with nothing happening beforehand (like ocean cooling) then we can try Laplace transforms.

We will finish the module with two examples of the application of Laplace transforms for ODEs

Cauchy's Theorem and the Calculus of Residues

These involve the concept of analytic functions of a complex variable, f(z), which we will loosely define to be functions that are differentiable in some simple domain in the complex plane which is bounded by a simple closed curve Γ.



We can test whether a function $F(z) = u(z) + i\, v(z)$ is *analytic* by testing whether it satisfies the Cauchy-Riemann equations:

$$\partial u / \partial x = \partial v / \partial y$$

$$\partial v / \partial x = - \partial u / \partial y$$

Cauchy's Theorem states that if $f(z)$ is analytic on and inside such a Γ, then

$$\int_{\Gamma} f(z)\, dz = 0$$

e.g. exp(z) is analytic everywhere.

$$\exp(z) = \exp(x + i y)$$
$$= \exp(x) \exp(iy)$$

exp(z) is analytic everywhere.

$$\exp(z) = \exp(x + i y)$$
$$= \exp(x) \exp(iy) = \exp(x) (\cos y + i \sin y)$$

So $u = \exp(x) \cos y$, $v = \exp(x) \sin y$

$\partial u / \partial x = \exp(x) \cos y$, $\partial v / \partial y = \exp(x) \cos y$;

$\partial v / \partial x = \exp(x) \sin y$, $\partial u / \partial y = - \exp(x) \sin y$ QED

Now suppose f(z) has some particular kinds of singularity. If

$$\lim_{z \to z_0} (z - z_0)\, f(z)$$

converges to an analytic function, we say that f(z) has a *simple pole* at z_0.

Eg sin(z)/(z - i) has a simple pole at z = i.

This is extended as follows: If

$$\lim_{z \to z_0} (z - z_0)^k\, f(z)$$

converges to an analytic function, we say that f(z) has a pole of order k at z_0.

Eg $\sin(z)/(z- i)^3$ has a pole of order 3 at z = i.

Suppose in the figure above, f(z) has n poles, z_1, z_2 ,.. z_n of order k_1, k_2 …k_n. We define the *Residue* of f(z) at one of these poles by:

$$\mathrm{Res}(f(z_j)) = 1/(k_j - 1)!\ \lim_{z \to z_j}\ \partial^{\,kj-1} / \partial z^{\,kj-1} [(z - z_j)^{kj} f(z)]$$

Thus the residue of $\sin(z)/(z- i)^3$ at z = i is given by

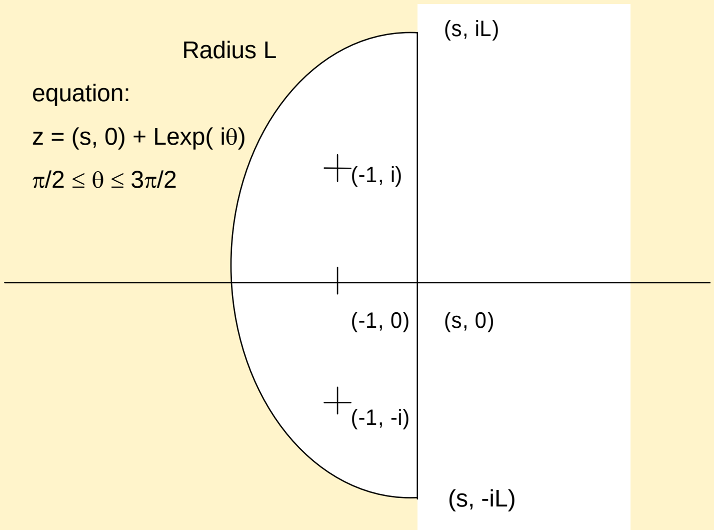
$$1/2!\quad \lim_{z \to i}\ \partial^{\,2} / \partial z^{\,2}\ [(z - i)^3 \sin(z)/(z- i)^3] = -1/2 \sin(i)$$

The *Residue Theorem* says: if f(z) has *n poles, z_1, z_2 , z_n of order k_1, k_2 …k_n* inside Γ, then

$$\int_\Gamma f(z)\, dz = 2\pi i\, \Sigma\, \mathrm{Res}(f(z_j))$$

So *we can calculate the integral of an analytic function*, or one with poles as singularities *by adding up the residues* – remarkable result with valuable applications to DEs.

Eg (coming) f has poles at (-1, i), (-1, -i)



Problem 1:

Solve $d^2u/dx^2 + (1/x)\, du/dx + u = 0$

with Boundary Conditions $u(0) = 1$, $du/dx(0) = 0$.

Henceforth write u' for du/dx, u'' for d^2u/dx^2 .

Problem 1:

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Henceforth write u' for du/dx, u'' for d^2u/dx^2 .

We should recognise the ODE as Bessel's equation for the simple n=0 case, so we know the answer; $u(x) = J_0(x)$, which does satisfy the BCs.

We will find a solution by Laplace transforms (from the table: $\mathcal{L}[J_0(x)](s) = (1+s^2)^{-1/2}$).

We need a further 'operational formula', so called because we need to check that it is valid in specific cases.

Lemma:

$$\mathcal{L}[\, x\, f(x)\,](s) = -\, d/ds\, \mathcal{L}[\, f(x)\,](s)$$

Proof:

$$d/ds\, \mathcal{L}[\, f(x)\,](s) = d/ds \int_0^\infty \exp(-sx)\, f(x)\, dx$$

which we will calculate formally, by considering

$$\lim_{\delta \to 0} \frac{\int_0^\infty \exp(-(s+\delta)x)\, f(x)\, dx - \int_0^\infty \exp(-sx)\, f(x)\, dx}{\delta}$$

$$= \lim_{\delta \to 0} \int_0^\infty (1/\delta)\, \{\exp(-(s+\delta)x) - \exp(-sx)\}\, f(x)\, dx$$

But exp(-sx) is differentiable. So the integral is

$$= \int_0^\infty - x \exp(-sx)\, f(x)\, dx \quad \text{QED}$$

Also recollect that

$$\int_0^\infty \exp(-sx)\, f\,'(x)\, dx = s\, \mathcal{L}[f](s) - f(0)$$

and

$$\int_0^\infty \exp(-sx)\, f\,''(x)\, dx = s^2\, \mathcal{L}[f](s) - s\, f(0) - f\,'(0)$$

Today

Finish example 1

Convolutions: convolution theorem for Fourier and Laplace transforms

Example 2

So apply the Laplace transform to:

$x u'' + u' + x u = 0$, where we have multiplied through by x ($\neq 0$) and get

$$-d/ds \{ s^2 \mathcal{L}[u](s) - s\,u(0) - u'(0) \} + s\,\mathcal{L}[u](s) - u(0) - d/ds \{ \mathcal{L}[u](s) \}$$
$$= 0$$

Substitute $u(0) = 1$, $u'(0) = 0$ and differentiate:

$$-2s\,\mathcal{L}[u] - s^2\, d/ds\, \mathcal{L}[u] + 1 + s\,\mathcal{L}[u](s) - 1 - d/ds \{ \mathcal{L}[u](s) \} = 0$$

Rearranging:

$$(1 + s^2)\, d/ds\, \mathcal{L}[u] + s\,\mathcal{L}[u] = 0$$

So

$$1/\mathcal{L}[u]\; d/ds\, \mathcal{L}[u] = -s/(1 + s^2)$$

Which we can integrate:

$$\int 1/\mathcal{L}[u]\; d\mathcal{L}[u] = -\int s/(1 + s^2)\, ds + \text{constant}$$

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so

$$\ln(\mathcal{L}[u]) = -\tfrac{1}{2} \ln(1 + s^2) + \ln A \quad \text{(say)}$$

or

$$\mathcal{L}[u] = A/(1 + s^2)^{1/2} \qquad \text{(Put } A = 1\text{)}$$

NB singularities (not poles!) at $s = (0, i), (0, -i)$

$1/(1 + s^2)^{1/2}$ is an *integrable singularity*.

We have just found the Laplace transform of $J_0(x)$! Which is as far as we can go with this example, except to note that the inverse transform theorem provides a way to compute $J_0(x)$.

Problem 2:

Solve u'' + 2 u' + 2 u = f(x), for $x \geq 0$ (e.g. forced, damped oscillator)

with Boundary Conditions u(0) = 0, u'(0) = 0.

which is a 2nd order inhomogeneous ODE which standard methods will solve. We use it to illustrate calculating inverse Laplace transforms.

Taking LTs as before:

$s^2 \mathcal{L}[u](s) - s\, u(0) - u'(0) + 2 \{s \mathcal{L}[u](s) - u(0)\} + 2 \mathcal{L}[u](s) = \mathcal{L}[f]$

Substitute the Boundary Conditions and collect:

$$\mathcal{L}[u] (s^2 + 2 s + 2) = \mathcal{L}[f]$$

so

$$\mathcal{L}[u] = 1/(s^2 + 2 s + 2)\, \mathcal{L}[f]$$

Diversion: Convolution lemma.

So by the convolution lemma, u is the convolution of the inverse transforms of $1/(s^2 + 2s + 2)$ and $\mathcal{L}[f]$; but $\mathcal{L}^{-1}\mathcal{L}[f] = f(x)$.

So it remains to calculate the inverse transform of $1/(s^2 + 2s + 2)$, using the inverse transform:

$$u(x) = (1/2\pi i) \lim_{L \to \infty} \int_{s - iL}^{s + iL} \exp(zx)/(z^2 + 2z + 2)\, dz$$

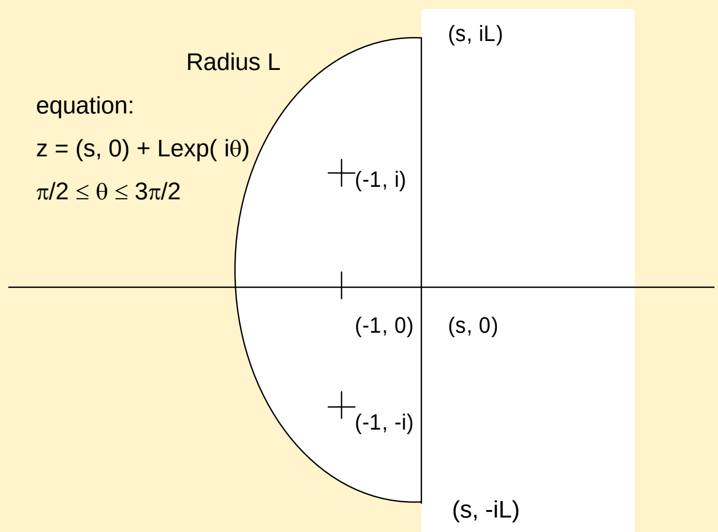
where we have exchanged s for a complex z in the integrand. Before we choose a closed contour to integrate around, we need to consider the singularities of $1/(z^2 + 2z + 2)$.

Find the zeros of $(z^2 + 2z + 2)$ by setting $(z^2 + 2z + 2) = 0$. Then

$$z = -1 \pm 1/2 \sqrt{(-4)} = -1 \pm i$$

So $1/(z^2 + 2z + 2)$ has *simple poles* at $z = -1 + i, -1 - i$.

So choose a contour as follows:



So the residues theorem says that the integral around this contour of

$\exp(zx)/(z^2 + 2z + 2) = \exp(zx)/[(z + 1+i)(z + 1 - i)]$, which is analytic except at the poles,

$= 2\pi i$ sum of the residues at the two poles

$$= 2\pi i \left\{ \lim_{z \to -1-i} (z+1+i)\exp(zx)/[(z+1+i)(z+1-i)] + \lim_{z \to -1+i} (z-1-i)\exp(zx)/[(z+1+i)(z+1-i)] \right\}$$

$$= 2\pi i \{ \exp((-1-i)x)/(-2i) + \exp((-1+i)x)/(2i)$$

$$= 2\pi i \ \exp(-x)\{ -\exp(-ix) + \exp(ix) \} = 2\pi i \exp(-x) \sin x$$

It remains to consider the integral around the semicircle, radius L,

$z = (s, 0) + L(\cos\theta + i \sin\theta)$, for $\pi/2 \le \theta \le 3\pi/2$.

In the integrand, $\exp(zx)/(z^2 + 2z + 2)$, $1/|(z^2 + 2z + 2)| \to 0$ as $L \to \infty$.

So the critical term is

$\exp(zx)$

Now for $\pi/2 \le \theta \le 3\pi/2$, *$\cos\theta < 0$*, $x \ge 0$ by assumption, so for L sufficiently large $(s + L\cos\theta)x < 0$.

So $\exp(zx) \to 0$ as $L \to \infty$ because the real part of zx is negative.

Thus integrand $\to 0$ as $L \to \infty$

and the contribution to the contour integral = 0.

So we are left with:

$\mathcal{L}^{-1}$ [1/(z^2 + 2 z + 2)]

$$= (1/2\pi i) \lim_{L\to\infty} \int_{s - iL}^{s + iL} \exp(zx) / (z^2 + 2z + 2)\, dz = (1/2\pi i)\, 2\pi i \exp(-x) \sin x$$

$$= \exp(-x) \sin x$$

And so using the convolution theorem

$$u(x) = \int_0^x \exp(-(x - y)) \sin(x - y) f(y)\, dy$$

is the solution to the problem. Remember the integration is to x, not ∞.

Note how f(x) has been incorporated into the solution for u(x).

Quick diversion:

Convolution formula for Laplace Transformable functions

To be able to take Laplace transforms, we require f(x) and g(x) are 0 for x < 0

$$f * g(x) = \int_0^\infty f(x - y) g(y)\, dy = \int_0^x f(x - y) g(y)\, dy$$

See http://en.wikipedia.org/wiki/Convolution

because f(x – y) = 0 for x – y < 0, or y > x.

Then the convolution theorem is:

$$\int_0^\infty f * g(x) \exp(-\nu x)\, dx = \left\{ \int_0^\infty f(x) \exp(-\nu x)\, dx \right\} \left\{ \int_0^\infty g(x) \exp(-\nu x)\, dx \right\}$$

for x ≥ 0.

Proof: LHS is

$$\int_0^\infty \{ \int_0^x f(x - y)\, g(y)\, dy \} \; \exp(-\nu x)\, dx$$

We have to be careful since x now appears in the integrand and the endpoint. So we make use of the *Heaviside* function

$$H(x - a) = 0 \text{ for } x < a;$$

$$= 1 \text{ for } x \geq a.$$

So write LHS as

$$\int_0^\infty \{ \int_0^\infty f(x - y)\, g(y)\, H(x - y)\, dy \} \; \exp(-\nu x)\, dx$$

Now put $u = x - y$; so $du = dx$. $f(u)$ is zero for $u < 0$ or $x < y$, and

$$\exp(-\nu x) = \exp(-\nu [u + y]) = \exp(-\nu u)\, \exp(-\nu y)$$

so LHS is

$$\int_{u=-y}^\infty \{ \int_{y=0}^\infty f(u)\, g(y)\, H(u)\, \exp(-i \nu y)\, dy \} \; \exp(-i \nu u)\, du$$

But $H(u) = 0$ for $u < 0$, so LHS

$$= \{ \int_0^\infty f(u)\, \exp(-i \nu u)\, du \} \{ \int_0^\infty g(y)\, \exp(-i \nu y)\, dy \}$$

QED.

END OF MODULE!

Except...

TUTORIAL FRIDAY